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R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization

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Abstract: We conjecture a general formula for assigning R-charges and multiplicities for the chiral fields of all gauge theories living on branes at toric singularities. We check that the central charge and the dimensions of all the chiral fields agree with the information on volumes that can be extracted from toric geometry. We also analytically check the equivalence between the volume minimization procedure discovered in hep-th/0503183 and a-maximization, for the most general toric diagram. Our results can be considered as a very general check of the AdS/CFT correspondence, valid for all superconformal theories associated with toric singularities.

Keywords: AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

D3 branes living at conical Calabi-Yau singularities are a good laboratory for the AdS/CFT correspondence since its early days. The world-volume theory on the branes is dual to a type-IIB background of the form $AdS_5 \times H$, where H is the horizon manifold [1, 2]. Supersymmetry requires that H is a Sasaki-Einstein manifold. Until few months ago, the only known Sasaki-Einstein metrics were the round sphere S^5 and $T^{1,1}$, the horizon of the conifold. Recently, various infinite classes of new regular Sasaki-Einstein metrics were constructed [3 − 5] and named $Y^{p,q}$ and $L^{p,q,r}$. For infinite values of the integers p, q, r one obtains smooth Sasaki-Einstein manifolds. With the determination of the corresponding dual gauge theory (see [6] for the $Y^{p,q}$ manifolds and [7 − 9] for the $L^{p,q,r}$), new checks of the AdS/CFT correspondence were possible [10, 6, 11, 12, 7 − 9]. As well known, the central charge of the CFT and the dimension of some operators can be compared with the volumes of H and of some of its submanifolds. In particular, the a-maximization technique [13] now allows for a detailed computation of the relevant quantum field theory quantities. Needless to say, the agreement of the two computations is perfect.

The number of explicit metrics for Sasaki-Einstein horizons than can be used in the AdS/CFT correspondence is rapidly increasing. However, to demystify a little bit the importance of having an explicit metric, we should note that all relevant volumes are computed for calibrated divisors. This means that these volumes can be computed without actually knowing the metric. There exist moreover a beautiful geometrical counterpart of the a-maximization [13]: this is the volume minimization proposed in [14] for determining the Reeb vector for toric cones. This procedure only relies on the vectors defining the toric fan. This suggests that with a correspondence between toric diagrams and gauge theories, many checks of the AdS/CFT correspondence can be done without an explicit knowledge of the metric. It is the purpose of this paper, indeed, to show that the knowledge of the toric data is sufficient to determine many properties of the dual gauge theory and to perform all the mentioned checks, for every singularity.

The precise correspondence between conical Calabi-Yau singularities and superconformal gauge theories is still unknown. However, a remarkable progress has been recently made for the class of Gorenstein toric singularities. The brane tiling (dimers) construction [15], an ingenious generalization of the Brane Boxes [16, 17], introduces a direct relation between an Hanany-Witten realization [18] for gauge theory and the toric diagram. In particular, from the quiver associated with a non-chiral superconformal gauge theory one can determine the dual brane tiling configuration, a dimer lattice. It is then possible to associate a toric diagram with each of these lattices, identifying the dual Calabi-Yau. The inverse process (to associate a gauge theory with a given singularity) is more difficult. However, for the mentioned checks of the AdS/CFT correspondence, we don't really need the full quiver description of the gauge theory. We just need to know the R-charges and the multiplicities of chiral fields. In this paper, elaborating on existing results in the literature [19, 7, 6, 9], we propose a general assignment of charges and multiplicities for the gauge theory dual to a generic Gorenstein singularity. This assignment is made using only the toric data of the singularity. We then compare the result of a-maximization with that of volume minimization showing that the two procedures are completely equivalent. This agreement is remarkable. We have two different algebraic procedures for computing the R-symmetry charges of the fields and the volumes. The first is based on the maximization of the central charge [13]. The second one can be efficiently encoded in a geometrical minimization procedure for determining the Reeb vector [14]. The two procedures deal with different test quantities (the R-charges on one side and the components of the Reeb vector on the other) and with different functions to be extremized. However, we will show that, with a suitable parametrization, the two functions (a and the inverse volume) are equal, even before extremization.

The agreement of results in the gauge theory and the supergravity side can be regarded as a general non-trivial check of the AdS/CFT correspondence, valid for all the theories living on branes at toric singularities.

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The paper is organized as follows. In section 2 we briefly review the general features of the gauge theories dual to conical singularities. In section 3 we propose the assignment of R-charges and multiplicities for the gauge theory in terms of geometrical data. In section 4 we show the equivalence of the a-maximization and the volume minimization. Section 5 contains several examples based on known gauge theories and various observations. In particular, as a by product of our analysis, we discuss in detail the case of the manifolds $X^{p,q}$ introduced in [20] whose general analysis was missing in the literature. We also make some observations on the identification of fields using the brane tiling technology. Finally, the appendix contains the proofs of various results that are too long and boring for the main text.

2. Generalities about the gauge theory

We consider N D3-branes living at a conical Gorenstein singularity. The internal manifold is a six-dimensional symplectic toric cone; its base, or horizon, is a five-dimensional Sasaki-Einstein manifold H [1, 2]. As well known, the $\mathcal{N} = 1$ gauge theory living on the branes is superconformal and dual in the AdS/CFT correspondence to the type-IIB background $AdS_5 \times H$. The gauge theory on the world-volume of the D3 branes is not chiral and represents a toric phase [21], where all gauge groups have the same number of colors N and the only matter fields are bi-fundamentals. By applying a Seiberg duality we can obtain a different theory that flows in the IR to the same CFT. If we dualize a gauge group with number of flavors equal to $2N$ we remain in a toric phase where all gauge groups have number of colors N. In this process the number of gauge groups remains constant but the number of matter fields changes. In a toric phase the following relation between the number of gauge groups F, the number of chiral fields E and the number of terms in the superpotential V

$$V - E + F = 0 \tag{2.1}$$

is valid [15]. Indeed for a gauge theory living on branes placed at the tip of toric CY cone, one can extend the quiver diagram, drawing it on a torus T^2. The dual graph, known as the brane tiling associated with the gauge theory [15], has F faces, E edges and V vertices and it is still defined on a torus. The previous formula then follows from the Euler formula for a torus [15].

We can assign an R-charge to all the chiral fields. The most general non-anomalous R-symmetry is determined by the cancellation of anomalies for each gauge group and by the requirement that each term in the superpotential has R-charge 2. This would seem to imply $F + V = E$ linear conditions for E unknowns with an unique solution. However, in the cases we are interested in, not all the conditions are linearly independent. This is reflected by the fact that the R-symmetry can mix with all the non anomalous U(1) global symmetries. We can count the number of global non-anomalous U(1) symmetries from the number of massless vectors in the AdS dual. Since the manifold is toric, the metric has three U(1) isometries. One of these (the Reeb one) corresponds to the R-symmetry while the other two are related to non-anomalous global U(1)s. Other gauge fields in AdS come from the reduction of the RR four form on the non-trivial three-spheres in the horizon manifold H. The number of three-cycles depends on the topology of the horizon, and, as we will review soon, can be computed using the toric data of the singularity. In the supergravity literature the vector multiplets obtained from RR four form are known as the Betti multiplets. On the gauge theory side, these gauge fields correspond to baryonic symmetries.

At the fixed point, only one of the possible non-anomalous R-symmetry enters in the superconformal algebra. It is the one in the same multiplet as the stress-energy tensor. The actual value of the R-charges at the fixed point can be found by using the a-maximization technique [13]. As shown in [13], we have to maximize the a-charge [22]

$$a(R) = \frac{3}{32}(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R)\,. \tag{2.2}$$

It is not difficult to show that the absence of anomalies implies $\mathrm{Tr}\,R = 0$ so that we can equivalently maximize $\mathrm{Tr}\,R^3$.

The results of the maximization give a complete information about the values of the central charge and the dimensions of chiral operators at the conformal fixed point. These can be compared with the prediction of the AdS/CFT correspondence [23, 24]. The first important point is that the central charge is related to the volume of the internal manifold [23]

$$a = \frac{\pi^3}{4\mathrm{Vol}(H)}\,. \tag{2.3}$$

Moreover, recall that in the AdS/CFT correspondence a special role is played by baryons. The gravity dual describes a theory with $\mathrm{SU}(N)$ gauge groups. The fact that the groups are $\mathrm{SU}(N)$ and not $\mathrm{U}(N)$ allows the existence of dibaryons. Each bi-fundamental field Φ_α^β gives rise to a gauge invariant baryonic operator

$$\epsilon^{\alpha_1\cdots\alpha_N}\,\Phi_{\alpha_1}^{\beta_1}\cdots\Phi_{\alpha_N}^{\beta_N}\,\epsilon_{\beta_1\cdots\beta_N}\,.$$

It is sometime convenient to think about the baryonic symmetries as non-anomalous combinations of $\mathrm{U}(1)$ factors in the enlarged $\prod \mathrm{U}(N)$ theory. In the AdS dual the baryonic symmetries correspond to the reduction of the RR four form and the dibaryons are described by a D3-brane wrapped on a non-trivial three cycle. The R-charge of the i-th field can be computed in terms of the volume of the corresponding cycle Σ_i using the formula [24]

$$R_i = \frac{\pi\mathrm{Vol}(\Sigma_i)}{3\mathrm{Vol}(H)}\,. \tag{2.4}$$

3. Geometrical formulae for the R-charges

In this section we propose a general formula for the R-charges and the multiplicity of chiral fields based only on the toric data.[1] This proposal is the natural combination of existing results [19, 7, 6, 9] and it is substantially implicit in previous papers on the subject. It is based on a formula for multiplicities first derived using mirror symmetry [19]. The same proposal appeared for the case of $L^{p,q;r,s}$ manifolds in [7], under the name of "folded quiver".

[1]For the necessary elements of toric geometry see [25] and the review part of [26].

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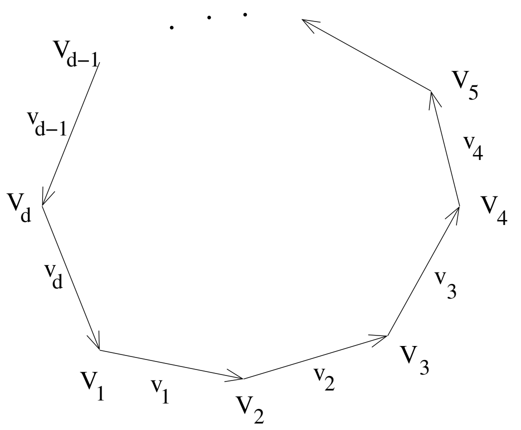

Figure 1: The convex polygon P.

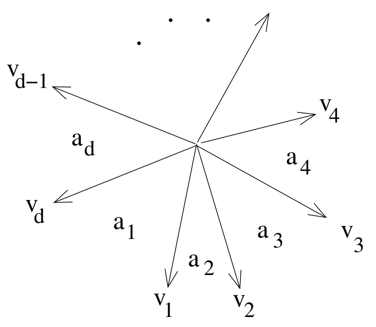

Figure 2: The (p, q) web for P.

The fan associated with a six-dimensional symplectic toric cone is generated by d integers primitive vectors in $\mathbb{R}^3$, which we call V_i, $i = 1, 2 \dots d$. When the cone is a Calabi-Yau manifold, we can perform an $\mathrm{SL}(3, \mathbb{Z})$ transformation to put the first coordinates of the V_i's equal to 1. The intersection of the fan with the plane of points having the first coordinate $x = 1$ is thus a convex polygon P, called toric diagram, and we shall call the vectors associated to its sides v_i, $i = 1, 2 \dots d$, as in figure 1. In figure 2 we draw the corresponding (p, q) web: the vectors v_i have the same length than the edges of the polygon P.[2] Let us also define the symbol:

$$\langle w_i, w_j \rangle \equiv \det(w_i, w_j) \tag{3.1}$$

that is the determinant of the matrix with w_i and w_j as first and second line respectively, where w_i and w_j are two vectors in the plane of P. This is the oriented area of the parallelogram generated by w_i and w_j.

Some of the data of the gauge theory can be extracted directly from the geometry of the cone. In particular, there exist simple formulae for the number of gauge groups F and the total number of chiral bi-fundamental fields E [19, 6]

$$\begin{aligned} F &= 2\mathrm{Area}(P) \\ E &= \frac{1}{2} \sum_{i,j} |\langle v_i, v_j \rangle| \,. \end{aligned} \tag{3.2}$$

Notice that the expression for E refers to a particular toric phase of the gauge theory. The number of toric phases of a theory can be large; hopefully, the value of E in formula (3.2) refers to the phase with the minimal number of fields.

The R-charges and the multiplicity of fields with given R-charge are more difficult to determine. Here we make a proposal based on the following general observations. Each

[2] With a little abuse of notation we call v_i both the sides of P and the vectors of the (p, q) web. In fact they differ only by a rotation of 90^o. When some of the sides of the polygon P pass through integer points, that is for singular horizons, we should consider more complicated (p, q) webs; here we are ignoring such subtleties. We claim that this does not affect the process of a-maximization, since it is equivalent to setting to zero the charges b_i associated with integers points on the sides of P (see subsection 5.6).

chiral field is associated with a dibaryon and, consequently, with a supersymmetric three cycle in the horizon H. The cone over this cycle is a divisor in the symplectic cone $C(H)$. Each edge V_i in the fan determines a divisor D_i and the collection of the D_i, subject to some relations, is a complete basis of divisors for $C(H)$ [25]. We can therefore associate a type of chiral field to each vector V_i [9] and assign it a trial R-charge a_i. It is important to stress that more than one chiral field is associated with a single divisor: as pointed out in [9], a D3 brane wrapped on the cycle Σ may have more than one supersymmetric vacuum and each of these corresponds to a different bi-fundamental but with the same R-charge. As shown in [14] the volumes of the base three cycles Σ_i of the divisors D_i satisfy the relation

$$\sum_{i=1}^{d} \mathrm{Vol}(\Sigma_i) = \frac{6}{\pi}\mathrm{Vol}(H) \tag{3.3}$$

which implies, using formula (2.4), $\sum_{i=1}^{d} a_i = 2$. In general, these d fields will not exhaust all the different types of chiral fields. We expect the existence of other dibaryons obtained from divisors which are linear combinations of the D_is. The R-charges of the corresponding fields will not be independent but they will be determined as a linear combination of the a_is. Indeed, we claim that the a_is parametrize the most general R-symmetry.[3] The number of independent parameters in the trial R charge is equal to the number of global U(1) symmetries. We always have two global symmetries from the toric action and a number of baryonic symmetries equal to the number of three cycles. As shown in [14], the latter is equal to $d-3$; each baryonic symmetry B_a is indeed associated with a linear relation among the edges V_i

$$\sum_{i=1}^{d} B_i^a\, V_i = 0 \tag{3.4}$$

and there are exactly $d-3$ such relations. In conclusion, we have a total number of $d-1$ global U(1) symmetries which matches the number of independent parameters a_i.

Collecting all these pieces of information, we can propose the following assignments of R-charges and multiplicities for the chiral fields in the gauge theory:

- Associate with each edge vector V_i a chiral field with trial R-charge a_i, with the constraint,

$$\sum_{i=1}^{d} a_i = 2 \tag{3.5}$$

- Call C the set of all the unordered pairs of vectors in the (p, q) web; we label an element of C with the ordered indexes (i, j), with the convention that the vector v_i can be rotated to v_j in the counter-clockwise direction with an angle $\leq 180^o$. With our conventions $|\langle v_i, v_j\rangle| = \langle v_i, v_j\rangle$. Associate with any element of C the divisor

$$D_{i+1} + D_{i+2} + \cdots + D_j \tag{3.6}$$

[3]For a generalization of this sentence to singular horizons see subsection 5.6.

and a type of chiral field in the field theory with multiplicity $\langle v_i, v_j\rangle$ and R-charge equal to $a_{i+1} + a_{i+2} + \cdots + a_j$. The indexes i, j are always understood to be defined modulo d. For example in figure 2 the field associated to the pair $(d, 3)$ has R-charge $a_1 + a_2 + a_3$ and multiplicity $\langle v_d, v_3\rangle$. The total number of fields is the sum of all the multiplicities:

$$E \equiv \sum_{(i,j)\in C} |\langle v_i, v_j\rangle| \tag{3.7}$$

and thus reproduces formula (3.2).

More generally, we can assign global symmetry charges to all the fields. The algorithm is very similar to that for R-charges:

- Assign global charges a_i to the fields corresponding to vertices V_i. The only difference is that now a_i satisfy the relation:

$$\sum_{i=1}^{d} a_i = 0\,. \tag{3.8}$$

- The global charges of composite chiral fields are then: $a_{i+1} + a_{i+2} \cdots + a_j$ for the fields corresponding to (i, j) in C.

With a small abuse of notation, we will use the same letter a_i for R and global symmetries; in the first case they satisfy $\sum_{i=1}^{d} a_i = 2$, while in the latter $\sum_{i=1}^{d} a_i = 0$.

Note that with the assignment (3.8) we parametrize all the possible $d-1$ global symmetries, the $d-3$ baryonic ones and the two flavor ones. We can explicitly identify the baryonic symmetries as follows. As shown in [9], the chiral fields associated with the edges V_i have a charge under the baryonic symmetry B_a equal to the coefficient B_i^a in the linear relations (3.4). Notice that the baryonic charges of the fields associated with the edges V_i sum up to zero

$$\sum_{i=1}^{d} B_i^a = 0 \tag{3.9}$$

and therefore satisfy eq. (3.8) as a consequence of the Calabi-Yau condition; the latter requires that all the vectors V_i lie on a plane which, in our conventions, means that the first coordinate of all V_i is 1. In conclusion, among the global symmetries, those satisfying also the constraint (3.4) (with $a_i = B_i^a$) are the baryonic ones, the remaining two (for which there is not a natural basis, being mixed with baryonic symmetries [9]) are the flavor ones.

We conjecture that for every Gorenstein toric singularity there exists a toric phase of the dual gauge theory where the R-charges and the multiplicities of all chiral fields can be computed with the algorithm above. This toric phase has generally the minimal number of chiral fields (3.2), as we have checked in many known cases. To be concrete look at figure 4 corresponding to $L^{p,q;r,s}$. There are six kinds of fields: the four with charge a_i, fields with charge $a_3 + a_4$ and others with charge $a_2 + a_3$, but there are not for instance fields with charge $a_1 + a_2$, since the region formed by v_4 and v_2 which includes a_1 and a_2 in the (p, q) web has always an angle greater than 180^o. Note that in general the number of different kinds of fields is $d(d-1)/2$, the number of elements of C. Note also that the R-charges of composite chiral fields can be written as sum of consecutive a_is; since P is convex the ordering of vectors v_i in the (p,q) web is always equal to the ordering of v_i in P.

With this assignment, we have a trial central charge a given by:

$$a = \frac{9}{32}\,\mathrm{tr}\,R^3 = \frac{9}{32}\left(F + \sum_{(i,j)\in C} |\langle v_i, v_j\rangle|(a_{i+1} + a_{i+2} + \cdots a_j - 1)^3\right). \tag{3.10}$$

Recall that F is the double area of the polygon P (3.2). The values of the R-charges a_i can be found by (locally) maximizing this formula. Note that this formula and the algorithm proposed above are obviously invariant under translations and $\mathrm{SL}(2,\mathbb{Z})$ transformations in the plane of P, since $\langle v_i, v_j\rangle$ are conserved, also in sign.[4]

We can make several checks of this proposal. First of all, it is easy to compare the proposal to the case where the quiver gauge theory is explicitly known. Several examples are discussed in section 5. In some cases, the fields and their multiplicity can be determined by using mirror symmetry; this was done for the toric delPezzo in [19] where the formula for the multiplicities based on the (p,q) web first appeared. The multiplicity of the fields associated with the edges V_i was computed in [9] and agrees with our proposal:

$$\text{multiplicity of fields } (i-1,i) \in C = \langle v_{i-1}, v_i\rangle = \det(V_{i-1}, V_i, V_{i+1}) \tag{3.11}$$

since the fields corresponding to the pair $(i-1,i)$ in C have R-charge a_i; we have used that the first coordinates of V_i are equal to 1. A proposal identical to ours was used in [7] to determine the gauge theory for the $L^{p,q,r}$ manifolds.

We can next study the consistency of our proposal with the general properties of the U(1) symmetries in our theories. First of all, we must have

$$\mathrm{Tr}\,G = 0\,. \tag{3.12}$$

where G is a general R-charge or global symmetry charge. In particular $\mathrm{Tr}\,R = \mathrm{Tr}\,B^a = 0$. The proof of this formula is relatively easy and is reported in the appendix. Another non trivial check of our proposal is the proof, reported in the appendix, that, for baryonic symmetries,

$$\mathrm{Tr}\,B_a^3 = 0\,. \tag{3.13}$$

This condition, which is true also for mixed baryonic symmetries, is a consequence of the vanishing of the cubic t'Hooft anomaly for a baryonic symmetry. This follows from the fact that on the stack of D3 branes in type-IIB the baryonic symmetries are actually gauged. The counterpart of this statement in the AdS dual is that cubic anomalies are computed from the Chern-Simons terms in the five dimensional supergravity and no such term can contain three vector fields coming from reduction of the RR four-form [27].

[4] If the determinant is -1 all the signs are reversed and so relative orientations do not change.

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The best check of the proposal is however the computation of the R-charges at the fixed point using a-maximization and the comparison with volumes of three cycles in H. Now that we have an algorithm to extract the field content of the gauge theory from the toric diagram, it is not difficult to write down an algorithm on a computer and check the agreement of a-maximization with Z-minimization on arbitrary large polytopes. The complete agreement of the a-maximization with the volume minimization of [14] will be discussed in details in the next section, where a general analytic proof will be given.

We finish this section by making some remarks about other toric phases of the same CFT with more chiral fields than the minimal phase presented above. In practical examples we often meet toric phases with the same trial central charge a than the minimal phase; these phases generally contain all the kinds of fields of the minimal phase, but with greater multiplicities. In fact there are other possible assignments of R-charges and multiplicities leading to the same a charge. For example, to each element in C we may assign two different types of chiral fields, one associated with the divisor

$$C_{i,j} = D_{i+1} + D_{i+2} + \cdots + D_j \tag{3.14}$$

with R-charge $a_{i,j} = a_{i+1} + \cdots a_j$, and a second one associated with the divisor[5]

$$\sum_{i=1}^{d} D_i - C_{i,j} = D_{j+1} + D_{j+2} + \cdots + D_i \tag{3.15}$$

with R-charge $a_{j+1} + \cdots a_i = 2 - a_{i,j}$. If we assign multiplicities $n_{i,j}$ and $\tilde{n}_{i,j}$ to the two types of fields with the constraint

$$n_{i,j} - \tilde{n}_{i,j} = |\langle v_i, v_j\rangle| \tag{3.16}$$

it is easy to see that the equations $\mathrm{Tr}\, R = \mathrm{Tr}\, B_a = \mathrm{Tr}\, B_a^3 = 0$ are still satisfied. Moreover the expression for the trial central charge a is unchanged. Indeed the contribution of the integers $n_{i,j}$ to the central charge cancels:

$$n_{i,j}(a_{i,j} - 1)^3 + \tilde{n}_{i,j}(1 - a_{i,j})^3 \equiv |\langle v_i, v_j\rangle|(a_{i,j} - 1)^3\,. \tag{3.17}$$

The formula (3.2) for the number of chiral fields is obviously no more satisfied. Each time a field is split and a new arbitrary integer $n_{i,j}$ is introduced, the total numbers of fields increase. Formula (3.2) is strictly valid for the minimal presentation. We do not expect that for all arbitrary choices of $n_{i,j}$ and pairs (i, j) there exists a non minimal toric phase with multiplicities of chiral fields described by this splitting mechanism, even though many known toric phases are characterized by multiplicities determined in this simple way. In fact all the examples of (non minimal) toric phases considered in this paper are described by this splitting mechanism, and it would be interesting to know whether this is true in general.

[5]Recall that, in our conventions, the indexes i, j are always defined modulo d.

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4. a-maximization is volume minimization

For the purposes of the AdS/CFT correspondence, the R-charges of the chiral fields have to be matched with the volumes of the three-cycles bases Σ_i of the corresponding divisors. In the previous section we proposed a formula for computing the R-charges and the trial central charge a directly from the toric diagram. Moreover in [14] it was shown that all the geometric information on the volumes can be extracted from the toric data, through the process known as volume minimization (or Z-minimization), without any explicit knowledge of the metric. The reason for that is the following: supersymmetric cycles are calibrated and the volumes can be extracted only from the Kahler form on the cone. Therefore now it is possible to compare directly R-charges in the gauge theory and volumes in the geometry, checking the correctness of the AdS/CFT predictions for every toric CY cone. In this section we discuss the equivalence of a-maximization and Z-minimization.

We start by reviewing the work of [14] and reducing their formulas in the plane containing the convex polygon P. The Reeb vector K of a symplectic toric cone can be expanded in a basis e_i for the T^3 effective action on the fiber:

$$K = \sum_{i=1}^{3} b_i e_i \tag{4.1}$$

where the vector of coordinates $b = (b_1, b_2, b_3)$ lives inside the toric fan of the cone. The Reeb vector is associated with an R-symmetry in the dual gauge theory; by varying the vector we change the R-symmetry by mixing it with the global symmetries. From the geometrical point of view, the variation of the Reeb vector changes the metric and the volumes. For only one choice of vector $\bar{b}$ there exists a Calabi-Yau metric for the cone. The vector $\bar{b}$ has $\bar{b}_1 = 3$ and can be determined through the minimization of a certain function Z of the variables b_2 and b_3 [14]. We rephrase this process in the plane containing P by writing $b = 3(1, x, y)$ and allowing the point $B \equiv (x, y)$ to vary inside the convex polygon P: note in fact that b is inside the fan. Define the functions:

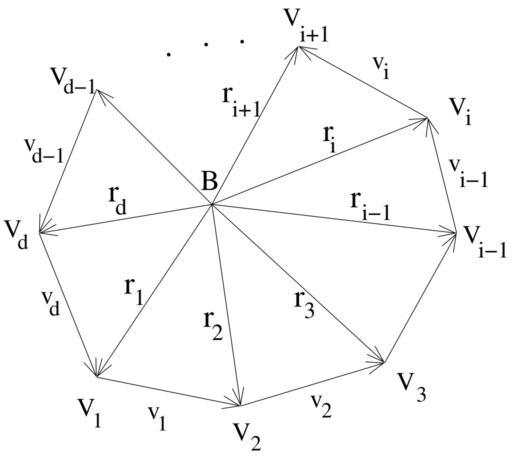

Figure 3: The Reeb vector as a point B inside the polygon P.

$$\mathrm{Vol}_{\Sigma_i}(x, y) = \frac{2\pi^2}{9} \frac{\langle v_{i-1}, v_i \rangle}{\langle r_{i-1}, v_{i-1} \rangle \langle r_i, v_i \rangle} \equiv \frac{2\pi^2}{9} l_i(x, y) \tag{4.2}$$

where r_i is the plane vector going from B to the vertex V_i (see figure 3).

As shown in [14], these are the volumes of the base three-cycles associated with the divisors $D_i, i = 1, \ldots, d$. Define also the function:

$$\mathrm{Vol}_H(x, y) = \frac{\pi}{6} \sum_{i=1}^{d} \mathrm{Vol}_{\Sigma_i}(x, y) \tag{4.3}$$

which determines the total volume of the horizon H. The two previous equations are just equations (3.25) and (3.26) of [14]. The function to minimize is just $\mathrm{Vol}_H(x,y)$[6] and the position of the minimum $(\bar{x},\bar{y})$ gives the Reeb vector $\bar{b}=3(1,\bar{x},\bar{y})$ for the CY cone. It was proved in [14] that such minimum exists and is unique.

The values of $\mathrm{Vol}_H(\bar{x},\bar{y})$ and $\mathrm{Vol}_{\Sigma_i}(\bar{x},\bar{y})$ at the minimum are the total volume of H and the volumes of Σ_i to be compared with the central charge a and the R-charges a_i of the field theory through the AdS/CFT relations (2.3) and (2.4). To facilitate this comparison we define the geometrical function:

$$a^{MSY}(x,y)=\frac{\pi^3}{4\mathrm{Vol}_H(x,y)} \tag{4.4}$$

and the functions:

$$f_i(x,y)=\frac{2l_i(x,y)}{\sum_{j=1}^d l_j(x,y)} \tag{4.5}$$

corresponding to the R-charges R_i through equation (2.4). The process of Z-minimization can be restated as a maximization of $a^{MSY}(x,y)$ with (x,y) varying in the interior of P.

On the other side of the correspondence we have the gauge theory with trial central charge a which is a function of the d variables a_i:

$$a(a_1,a_2,\dots a_d)=\frac{9}{32}\left(F+\sum_{(i,j)\in C}|\langle v_i,v_j\rangle|(a_{i+1}+a_{i+2}+\dots+a_j-1)^3\right). \tag{4.6}$$

We are considering a formal extension of the trial central charge to $\mathbb{R}^d$ defined by equation (3.10). This function has to be locally maximized with the constraint (3.5) (and $a_i>0$). To impose this constraint it is enough to introduce a Lagrange multiplier λ and to extremize the function:

$$a(a_1,a_2,\dots a_i)-\lambda(a_1+a_2+\dots+a_d-2). \tag{4.7}$$

By deriving with respect to a_i we get the conditions:[7]

$$\frac{\partial a}{\partial a_i}=\lambda \qquad i=1,\dots d. \tag{4.9}$$

If we call $\bar{a}_i$ the values of a_i at the local maximum, we have to prove that:

$$\begin{aligned} a^{MSY}(\bar{x},\bar{y}) &= a(\bar{a}_1,\bar{a}_2,\dots\bar{a}_d)\\ f_i(\bar{x},\bar{y}) &= \bar{a}_i \qquad i=1,\dots d. \end{aligned} \tag{4.10}$$

[6]This is the function Z in [14] up to a constant multiplicative factor.

[7]This means that the gradient of the extended function a in the local maximum is parallel to the vector $(1,\dots 1)$. So to extremize a it is enough to impose that the variations of a along the $d-1$ vectors S^a orthogonal to $(1,\dots 1)$ vanish:

$$\sum_{i=1}^d S_i^a\frac{\partial a}{\partial a_i}=0\,, \qquad \text{if } \sum_{i=1}^d S_i^a=0\,. \tag{4.8}$$

But note that, in the language of section 3, the space of S^a is just the space of the $d-1$ global symmetries (compare with (3.8)).

This is a highly non trivial check to perform: a-maximization and Z-minimization use different functions and different trial charges; it is not at all obvious why the result should be the same. First of all a-maximization is done on a total of $d-1$ independent trial parameters while the volume minimization is done only on two parameters (x, y). The trial central charge a is a cubic polynomial in a_i, whereas a^{MSY} is a rational function of (x, y). These parameters, in both cases, are somehow related to the possible global symmetries: the Reeb vector in the geometry is connected to R-symmetries of the gauge theory and changing the position of B in the directions x and y means adding to the R-symmetry the two flavor global symmetries.[8] In any case, the volume minimization is done by moving only in a two dimensional subspace of the set of global symmetries, while a-maximization is done on the entire space. Fortunately, as often claimed in the literature, a-maximization can be always performed on a two dimensional space of parameters related to flavor symmetries. Indeed, on general grounds, one can parametrize the trial R-symmetry as a contribution $R(X, Y)$ from the flavor symmetries plus a baryonic part

$$R = R(X, Y) + \sum_{a=1}^{d-3} h_a B_a \tag{4.11}$$

and the elimination of the variables h_a is simple: imposing that the derivatives of $\operatorname{tr} R^3$ with respect to h_a vanish, one gets the equations

$$\operatorname{Tr} R^2 B_a = 0\,. \tag{4.12}$$

These conditions read

$$\operatorname{Tr}\left(R(X, Y) + \sum_{b=1}^{d-3} h_b B_b\right)^2 B_a = 0 \tag{4.13}$$

which is a linear system of $d-3$ equations in the $d-3$ variables h_a. Linearity in h_a follows from the fact that the cubic mixed t'Hooft anomaly for baryonic symmetries is zero: $\operatorname{Tr} B_a^3 = 0$. So one can solve for h_a in function of X, Y and substitute into the trial charge (4.11); the central charge a is now a rational function only of X and Y. So we have reduced a-maximization to a maximization over a set of two parameters.

In the previous argument, the choice of a basis of flavor and baryonic symmetries in (4.11) was quite arbitrary. In our specific case we can choose a more natural parameterization for the two dimensional space over which to reduce a-maximization. This space is just the space of coordinates (x, y) of the plane where P lies: consider the map from $\mathbb{R}^2$ to $\mathbb{R}^d$ given by

$$\begin{aligned} f : (x, y) &\to (a_1, a_2, \dots a_d) \\ (x, y) &\to a_i = \frac{2 l_i(x, y)}{\sum_{j=1}^{d} l_j(x, y)} = f_i(x, y) \end{aligned} \tag{4.14}$$

[8] Recall that flavor symmetries are mixed with baryonic ones, so actually we are moving also in the space of baryonic symmetries.

We claim that the local maximum $(\bar{a}_1, \dots \bar{a}_d)$ of the a-maximization is found on the image $f(P)$ of the interior of P under this map. In fact it is not difficult to prove that the gradient of the trial central charge along the $d-3$ baryonic directions evaluated on $f(P)$ is always zero:

$$\sum_{i=1}^{d} B_i^a \frac{\partial a}{\partial a_i}_{|a_i=f_i(x,y)} = 0 \tag{4.15}$$

where B^a is a baryon charge and where the equality holds for every (x, y) in the interior of P. We give the general proof of (4.15) in the appendix. Note that equation (4.15) is completely equivalent to the condition (4.12) when the trial R-charge is evaluated with $a_i = f_i(x, y)$. Therefore we have clarified in which sense the baryonic symmetries decouple from the process of a-maximization.

At this point we have to compare the two functions $a^{MSY}(x, y)$ and the field theory trial central charge evaluated on the surface $f(P)$, which are two functions only of (x, y). Remarkably one discovers that they are equal even before maximization:

$$a(a_1, \dots a_d)_{|a_i=f_i(x,y)} = a^{MSY}(x, y) \tag{4.16}$$

for every (x, y) inside the interior of P. We give a general (still long) analytic proof of (4.16) in the appendix.

The proof of the equivalence of a-maximization and Z-minimization is now almost finished: we know that $a^{MSY}(x, y)$ has a unique maximum $(\bar{x}, \bar{y})$ inside the polygon P. In this point we have for the field theory a:

$$\frac{\partial f_i}{\partial x_h}(\bar{x}, \bar{y}) \left(\frac{\partial a}{\partial a_i}\right)_{|a_i=f_i(\bar{x},\bar{y})} = \frac{d}{dx_h} a(f_1(x, y), \dots f_d(x, y))_{|\bar{x},\bar{y}} = \frac{d}{dx_h} a^{MSY}(\bar{x}, \bar{y}) = 0 \tag{4.17}$$

where x_h, $h = 1, 2$ is x or y. So we see that, in the point $(\bar{x}, \bar{y})$, also the two vectors:

$$\frac{\partial f_i}{\partial x}(\bar{x}, \bar{y}) , \qquad \frac{\partial f_i}{\partial y}(\bar{x}, \bar{y}) \tag{4.18}$$

belonging to the space of global symmetries (since $\sum_i f_i = 2$) are orthogonal to the gradient of $a(a_1, \dots, a_d)$. Together with the $d-3$ baryon symmetries they span the whole $d-1$ space of global symmetries, thus proving (4.8) in the point $(\bar{x}, \bar{y})$. Therefore the extremum point for the trial central charge lies on the surface $f(P)$. One should also check that the Hessian matrix is negative definite to prove that this is a local maximum. The agreement between the volumes of Σ_i and the total volume with the R-charges $\bar{a}_i$ and the central charge in $(\bar{x}, \bar{y})$ follows immediately from the parametrization (4.14) and from (4.16).

5. Examples

In this section we provide various examples of our proposal using manifolds H where it is possible to determine the dual gauge theory explicitly. Needless to say, we find a remarkable agreement.

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5.1 The $Y^{p,q}$ manifolds

The superconformal theory dual to $AdS_5 \times Y^{p,q}$ has been determined in [6]. The cone $C(Y^{p,q})$ determines a polytope P with vertices

$$(0,0)\,, \qquad (1,0) \qquad (0,p) \qquad (-1,p+q) \tag{5.1}$$

with a (p,q) web given by the vectors

$$(0,-1)\,, \qquad (p,1) \qquad (q,1) \qquad (-p-q,-1) \tag{5.2}$$

With a toric diagram with four sides, we expect six different types of fields corresponding to the number of pairs (i,j). However, due to the non-abelian isometry of the manifolds, there is an accidental degeneration. Our proposal and the comparison with the known results is reported in table 1. using the notations of [6]. Recall that as usual $a_4 = 2 - a_1 - a_2 - a_3$. With this assignment we would perform the a-maximization on a three dimensional space of parameters. The enhanced global symmetry allows to reduce the parameter space to a two-dimensional one, as done in [6]. Indeed, in the a-maximization, R can mix only with abelian symmetries [13]; we still have $d-3 = 1$ baryonic symmetries, but only one U(1) flavor symmetry since the other is enhanced to SU(2). In any event, without knowing about the SU(2) symmetry we can perform a-maximization on three parameters and discover at the end that $a_2 = a_4$. In table 1, four fields are associated with the four edges of the fan. For $Y^{p,q}$ we obtain the fields Y,Z and two copies of the fields U with the same multiplicity p: they combine to give the SU(2) doublet U_α. The remaining two types of fields are associated with the divisors $D_2 + D_3$ and $D_3 + D_4$, they have multiplicity q and combine to give the doublets V_α.

$(i,j) \in C$	multiplicity	$\mathrm{U}(1)_R$	fields
$(4,1)$	p+q	a_1	Y
$(1,2)$	p	a_2	U
$(2,3)$	p-q	a_3	Z
$(3,4)$	p	a_4	U
$(1,3)$	q	$a_2 + a_3$	V
$(2,4)$	q	$a_3 + a_4$	V

Table 1: Charges and multiplicities for $Y^{p,q}$.

In the previous assignment $D_2 + D_3$ has been chosen instead of $D_4 + D_1$ because $\langle v_2, v_3\rangle > 0$. A similar argument applies to $D_3 + D_4$. It is also easy to check that all the toric phases of $Y^{p,q}$ described in [28] can be obtained in the way discussed at the end of section 3 (cfr table 1 in [28]).

5.2 The $L^{p,q;r,s}$ manifolds

The superconformal theory dual to $AdS_5 \times L^{p,q;r,s}$ has been determined in [7–9]. The cone $C(L^{p,q;r,s})$ determines a polytope P with vertices

$$(0,0)\,, \qquad (1,0) \qquad (P,s) \qquad (-k,q) \tag{5.3}$$

where k and P are determined through the Diophantine equation

$$r - ks - Pq = 0\,. \tag{5.4}$$

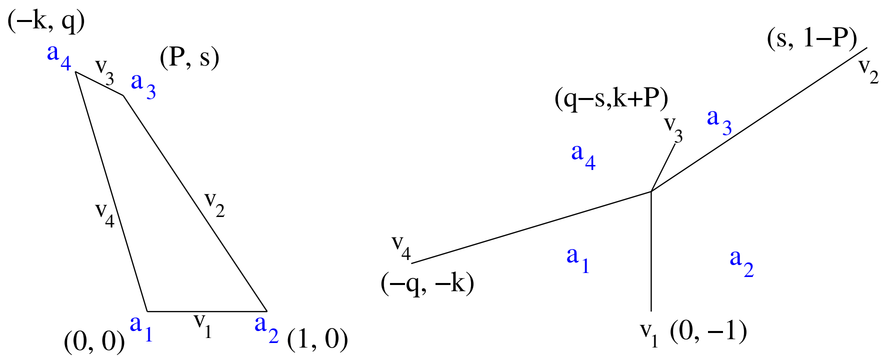


Figure 4: The toric diagram and (p,q) web for $L^{p,q;r,s}$.

Recall that $p+q=r+s$. As explained in [8], we can always choose $p \leq r \leq s \leq q$ without any loss of generality. The (p,q) web is given by the vectors

$$(0,-1)\,, \qquad (s,1-P) \qquad (q-s,k+P) \qquad (-q,-k)\,. \tag{5.5}$$

The toric diagram and (p,q) web for $L^{p,q;r,s}$ are reported in figure 4.

The toric diagram has four sides, and we expect six different types of fields corresponding to the number of pairs (i,j). In this case the isometry is $U(1)^3$ and we don't expect any degeneration. Our proposal and the comparison with the known results is reported in table 2 using the notations of [8]. Recall that as usual $a_4 = 2-a_1-a_2-a_3$. The same analysis was performed in [7].

$(i,j) \in C$	multiplicity	$U(1)_R$	fields
$(4,1)$	q	a_1	Y
$(1,2)$	s	a_2	$\tilde{W}$
$(2,3)$	p	a_3	Z
$(3,4)$	r	a_4	X
$(1,3)$	q-s	a_2+a_3	W
$(2,4)$	q-r	a_3+a_4	$\tilde{X}$

Table 2: Charges and multiplicities for $L^{p,q;r,s}$.

5.3 The $X^{p,q}$ manifolds

It is interesting to check the case of the manifolds $X^{p,q}$ discussed in [20]. These correspond to toric cones with five facets which can be blown down to the cones over $Y^{p,q}$. The corresponding gauge theory can be determined by an inverse Higgs mechanism [20]. The general assignment of R-charges and the a-maximization has not been performed in the literature except for particular p and q; therefore this model is an interesting laboratory.

The toric diagram is given by (see figure 5):

$$(1,p)\,, \qquad (0,p-q+1) \qquad (0,p-q) \qquad (1,0) \qquad (2,0) \tag{5.6}$$

and the (p,q) web is given by the vectors v_i (see figure 6):

$$(-q+1,1)\,, \qquad (-1,0) \qquad (-p+q,-1) \qquad (0,-1) \qquad (p,1) \tag{5.7}$$

With a toric diagram with five sides, we expect ten different types of fields corresponding to the number of pairs (i,j). Our proposal is reported in table 3. Recall that as usual

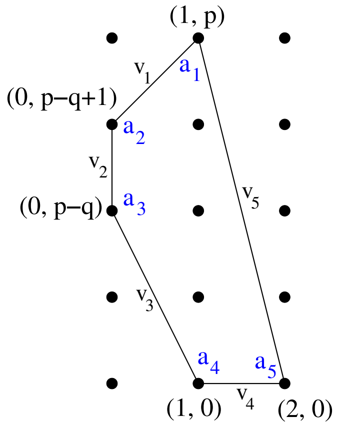


Figure 5: Toric diagram for $X^{p,q}$. We draw the case $X^{4,2}$.

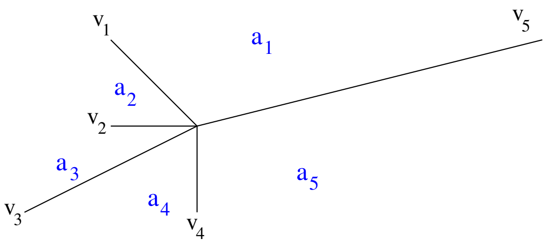


Figure 6: The (p, q) web for $X^{p,q}$.

$\sum_i a_i = 2$ for an R-symmetry. We have four independent parameters because there are now two baryonic symmetries.

We can explicitly determine the gauge theory and assign the R-charges to bi-fundamental fields. This can be done more efficiently using the brane tiling description of the $X^{p,q}$ theory. We refer to [15] for a detailed discussion of the brane tiling. Here we use the method we employed for the $L^{p,q;r,s}$ manifolds in [8]. The tiling for $X^{p,q}$ is pictured in figure 7. Similarly to $Y^{p,q}$ theories, the dimer configuration of $X^{p,q}$ can be obtained using only one column of n hexagons, and $m + 1$ consecutive cut hexagons. The horizontal identification has a shift $k = 1$, as for $Y^{p,q}$. The main difference is that for $X^{p,q}$ the last cut hexagon has a cut in the opposite direction than the other m cuts.

$(i, j) \in C$	multiplicity	$\mathrm{U}(1)_R$
$(5, 1)$	p+q-1	a_1
$(1, 2)$	1	a_2
$(2, 3)$	1	a_3
$(3, 4)$	p-q	a_4
$(4, 5)$	p	a_5
$(1, 3)$	p-1	$a_2 + a_3$
$(2, 4)$	1	$a_3 + a_4$
$(1, 4)$	q-1	$a_2 + a_3 + a_4$
$(5, 2)$	1	$a_1 + a_2$
$(3, 5)$	q	$a_4 + a_5$

Table 3: Charges and multiplicities for $X^{p,q}$.

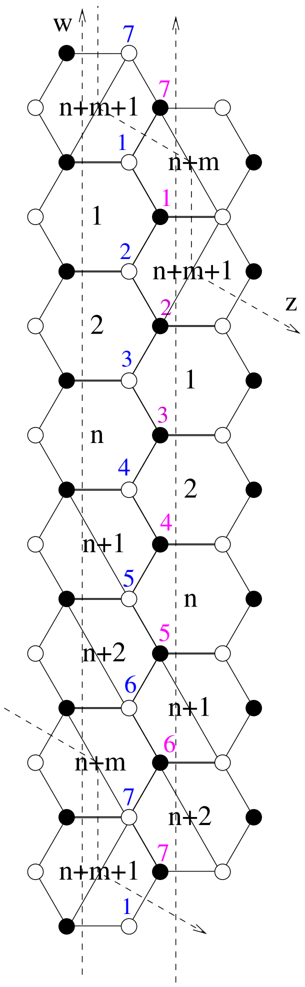
To fit the number of fields, gauge groups and superpotential terms for $X^{p,q}$ we must choose: $n = 2q - 1$, $m = p - q$. We report also the general form of the Kasteleyn matrix, with vertices numbered in the same way as in [8] (see figure 7). The determinant of K is then:

$$\det K = 1 + z + z^{-1} w^{n+m+1} + z^{-1} w^{n+m} + w^{\frac{n+1}{2}+m} + \cdots \tag{5.8}$$

where we have not been careful about signs and the omitted terms are powers of w with lower exponent. In the plane (z, w) one gets the toric diagram:

$$(0, 0)\,, \qquad (1, 0) \qquad (0, p) \qquad (-1, p + q) \qquad (-1, p + q - 1)\,. \tag{5.9}$$

Translating by $(0, -p)$, applying the $\mathrm{SL}(2, \mathbb{Z})$ transformation $((1, 0), (-p, -1))$ and translating by $(1, 0)$, one recovers the equivalent diagram (5.6). This shows that the dimer



$$
K = \left(\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 1 & w & 0 & 0 & 0 & 0 & z \\
2 & 1 & 1 & w & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 1 & w & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 1 & w & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 1-w & w & 0 \\
6 & 0 & 0 & 0 & 0 & 1 & 1-w & w \\
7 & wz^{-1} & -wz^{-1} & 0 & 0 & 0 & 1 & 1-w
\end{array}\right)
$$

Figure 7: The brane tiling for $X^{p,q}$ and the Kasteleyn matrix for $X^{p,q}$.

configuration reproduces the geometry. By comparing (5.9) with (5.1), it is manifest that the cone $C(X^{p,q})$ can be obtained by blowing up $C(Y^{p,q})$. Using the tiling in figure 7 and the algorithm described in [8][9] we can find the different types of fields and their distribution on the tiling in the general case. The agreement with our proposal given in table 3 is complete.

5.4 Assigning R-charges on the dimer: a general conjecture

In this subsection we propose, and check also on specific examples, a general conjecture to assign R-charges to chiral fields.

[9]the four independent symmetries are determined by the assignments v^1, v^2, v^3 as in appendix A.2 of [8] plus an assignment built as a second "cycle" starting from the cut hexagon at position $m+1$. An alternative and more general method for determining the distribution of R-charges on the dimer is described in subsection 5.4.

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In [15] it was suggested a natural one to one correspondence between auxiliary fields in the Witten sigma model associated with a quiver theory and perfect matchings of the dimer configuration. Recall that a perfect matching of a bipartite graph is a choice of links such that every white and black vertex is taken exactly once. We will concentrate on theories for which the multiplicities of the auxiliary fields in the associated Witten sigma model[10] corresponding to vertices of the toric diagram are all equal to one. That is we consider dimer configurations with only one perfect matching corresponding to each vertex of the toric diagram. This is always true in all known theories we considered and we think this may be true also in general.

In fact not only there exist many equivalent descriptions (dimer configurations) of the same physical theory, generally connected by Seiberg dualities, but there are also dimer configurations that do not have any AdS/CFT dual. As an example consider the dimers that can be built using only one column of n hexagons and m (consecutive) cut hexagons as in [8]. In that paper it was pointed out that, using an horizontal identification with shift $k = 1$, one can obtain the whole family of $Y^{p,q}$ theories with the choice $n = 2q$ and $m = p - q$. Note that n is always even, and the toric diagram of $Y^{p,q}$ is:

$$(0,0)\,, \qquad (1,0)\,, \qquad \left(0, \frac{n}{2} + m\right), \qquad (-1, n+m) \qquad \text{for } n \text{ even} \tag{5.10}$$

For these configurations the number of perfect matchings associated to any vertex V_i is always one, as it is easy to prove from the general expression of the Kasteleyn matrix reported in [8]. Moreover these configurations survive the test of the equivalence between a-maximization and Z-minimization.

Instead if we build tilings with an odd number N of normal hexagons and M cut hexagons (shift again k=1) we get surprising results. The toric diagram is now given by:

$$(0,0)\,, \qquad (1,0), \qquad \left(0, \frac{N-1}{2} + M\right), \qquad (-1, N+M) \qquad \text{for } N \text{ odd} \tag{5.11}$$

as one can see from the Kasteleyn matrix. Note that, up to auxiliary fields multiplicities, we get the same toric configuration if we choose:

$$N = n - 1 \qquad M = m + 1 \tag{5.12}$$

but with N odd there is a vertex of the toric diagram (precisely $(0, (N-1)/2 + M)$) having more than one corresponding perfect matching, as one can see again from the Kasteleyn matrix. Moreover it is easy to see that the theories corresponding to configurations $(N, M, k = 1)$ with N odd do not match the Z-minimization results of the corresponding toric diagrams. These quiver theories do not have a conformal fixed point satisfying the unitary bounds. In fact it is easy to convince oneself that they have only two[11] global symmetries instead of three U(1) symmetries of $Y^{p,q}$ (one of these U(1) is however enlarged

[10] not to be confused with the multiplicity of the "real" fields in the gauge theory associated to the vertex of the toric diagram.

[11] With N odd the cycle described in appendix A.2 of [8] to build the third charge do not cover all the cut hexagons.

to SU(2) for $Y^{p,q}$). The trial R-charges associated with some fields (those corresponding to the cuts of the hexagons) are zero and this violates the unitary bound since the corresponding gauge invariant dibaryon operators would have dimension zero. In this way, we have built an infinite family of quiver gauge theories, that can be represented with dimer configurations, but cannot have any geometric AdS/CFT dual. We analyzed some other cases of theories without a geometric dual by varying also k, and always found that such theories have at least a vertex of the toric diagram with number of perfect matchings associated greater than one. We conjecture that the request of having only a perfect matching corresponding to each vertex of the toric diagram is necessary for the existence of an AdS/CFT dual, but this statement should be further studied. In the following we only consider theories that satisfy such request.

Our conjecture is that it is possible to assign R charges (or global charges) once the perfect matchings corresponding to the vertices of the toric diagram are known.[12] The method is simple: assign R-charge (or global charge) a_i to the perfect matching corresponding to the vertex V_i of the toric diagram. The charges a_i satisfy (3.5) if they are R-charges or (3.8) if they are global charges. The (R-)charge of a link in the dimer configuration is then the sum of all (R-)charges of the perfect matchings (corresponding to vertices of the toric diagram) to which the link belongs.

We have checked in many known cases that this method works, also in different toric phases of the same theory. For phases with the minimal number of fields it reproduces our formula for the multiplicities of the different kinds of fields. For example it is not difficult to extract the perfect matchings associated to vertices of the $X^{p,q}$ theories from the Kasteleyn matrix reported in the previous subsection. And then one can check that the distribution of R-charges in the dimer obtained with the method proposed is a good distribution, that is one verifies that at every vertex the sum of R-charges is 2 (invariance of the superpotential) and for every face the sum of R-charges is equal to the number of edges minus 2 (beta functions equal to zero). We give other explicit examples of this method in the following subsection.

It would be interesting to check whether this method works in general. Obviously the invariance of the superpotential is guaranteed, since every perfect matching takes every vertex once and the sum of the a_i is 2. It would be necessary also to prove the condition for faces (zero beta functions).

In the toric phases with minimal number of fields the method for computing multiplicities of fields described in section 3 should hold. Every perfect matching is made up with $V/2$ links, where V, the number of vertices in the dimer configuration, is computed in minimal phases from the toric diagram as $V = E - F$. The method proposed in this subsection implies that there are exactly $V/2$ fields containing the charge a_1, and the same is true for every a_i, $i = 1, \ldots d$. Consistence with our formulas for computing multiplicities from the (p, q) web requires that the sum of multiplicities of all fields containing a_i is equal to $V/2$ independently from i. This is true and is proved in appendix A.

[12]We consider here smooth horizons for which the edges of the toric diagram do not pass through integer points, for an extension of this conjecture also to non smooth horizons see subsection 5.6.

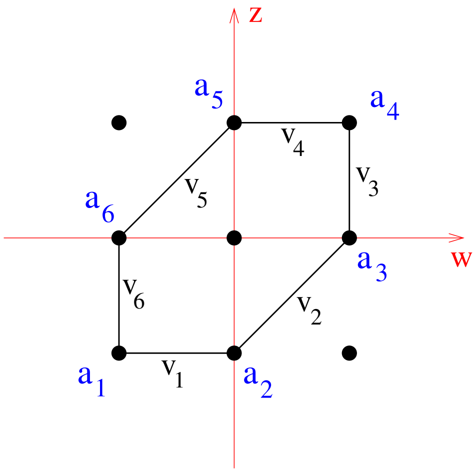

Figure 8: Toric diagram for dP_3.

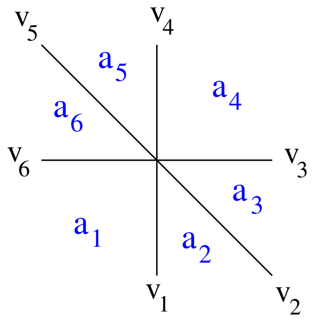

Figure 9: The (p, q) web for dP_3.

As a final remark, let us remind that in [15] it was discovered that a chiral field (a link in the dimer) in the gauge theory can be written as the product of all auxiliary fields associated to perfect matchings to which the field belongs, and not only to perfect matchings corresponding to vertices of the toric diagram. Hence we have claimed that only the perfect matchings associated to vertices are charged under R or global symmetries, whereas other perfect matchings have charges equal to zero.

5.5 The toric del Pezzo 3

In this subsection we consider the example of the theories associated with the complex cone over dP_3. This toric manifold is interesting since its toric diagram has six edges and four different toric phases are known. All the corresponding quivers are given in [15]. We draw in figures 8 and 9 the toric diagram and (p, q) web for dP_3; we also show the assignment of charges a_i in our conventions. Remember that for R-charges the sum of all a_i is equal to 2.

The area of the toric diagram is 3, and therefore the number of gauge groups is $F = 6$. Model I of dP_3 has 12 fields $E = 12$ and hence $V = 6$ terms in the superpotential. This model has the least number of fields among the toric phases of dP_3, in agreement with equations (3.2). We draw the dimer configuration for Model I in figure 10; we label the chiral fields X_i with numbers $i = 1, \dots 12$ typed in blue and vertices with letters $A, B, \dots F$. The identification of faces is as in [15].

To compute the R-charges of the theory we can use the method suggested in the previous subsection; first we have to know the perfect matchings associated to the vertices. A fast way to compute them is by writing the determinant of a modified Kasteleyn matrix:

$$K = \begin{array}{c|ccc} & B & D & F \\ \hline A & X_2 + wX_5 & X_3 - wzX_6 & X_4 + zX_1 \\ C & X_7 & -X_9 & -w^{-1}X_8 \\ E & -z^{-1}X_{12} & -X_{11} & X_{10} \end{array} \tag{5.13}$$

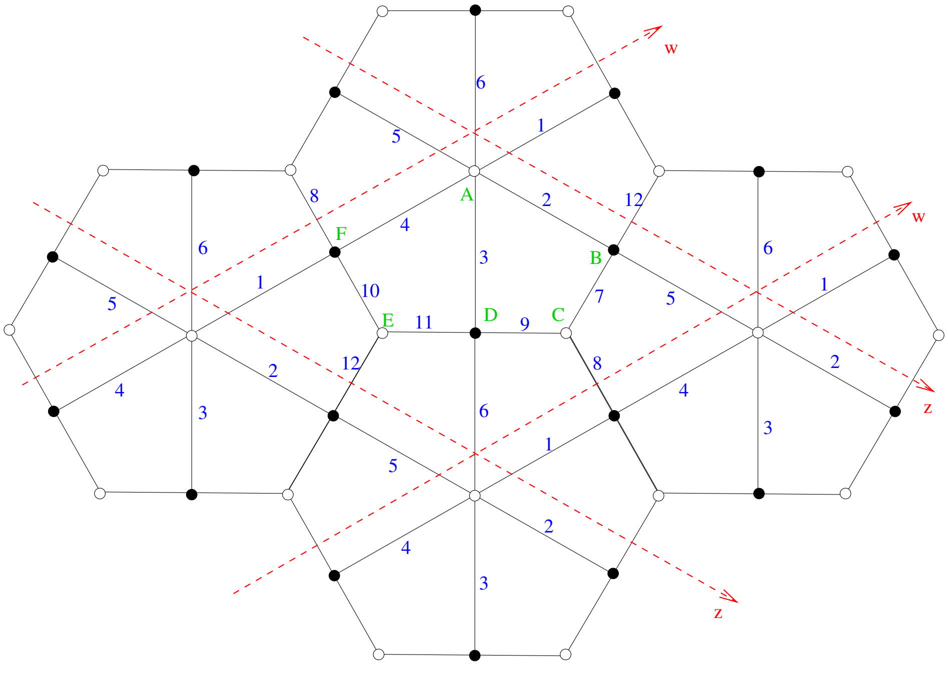

Figure 10: The dimer configuration for dP_3, Model I.

where we have written for every field not only the usual weight in function of w and z [15], but also the name of the field itself. Note that it is not necessary to be careful about signs. The coefficient of $w^i z^j$ in the expression of $\det K$ gives the perfect matching(s) associated with the point at position (i,j) in the plane of the toric diagram. So we find that the perfect matchings associated with the vertices are:

$$\begin{aligned}
a_1 &\to X_3 X_8 X_{12} \\
a_2 &\to X_4 X_9 X_{12} \\
a_3 &\to X_5 X_9 X_{10} \\
a_4 &\to X_6 X_7 X_{10} \\
a_5 &\to X_1 X_7 X_{11} \\
a_6 &\to X_2 X_8 X_{11}
\end{aligned} \tag{5.14}$$

where on the left we have written the R-charge associated with the vertex/perfect matching. We can then compute the R-charges of the fields X_i as described in the previous subsection by summing all the charges of the vertex perfect matchings to which a field X_i belongs. We thus get the following table for R-charges:

$$\begin{array}{cccccccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\
a_5 & a_6 & a_1 & a_2 & a_3 & a_4 & a_4+a_5 & a_1+a_6 & a_2+a_3 & a_3+a_4 & a_5+a_6 & a_1+a_2
\end{array} \tag{5.15}$$

We have found five independent trial R-charges (there is relation (3.5) among the a_i), and indeed it is not difficult to show that they are the correct ones, for example by writing the matrix C_{ij} as in appendix A.2 of [8].

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Note that the multiplicities (equal to 1 for dP_3) and the kinds of different fields just found are in agreement with the general formula we propose in this paper. So we recognize in Model I the minimal toric phase of dP_3 for which the formulae proposed in this paper strictly hold.

There are three other phases of dP_3 with more than 12 fields. We have performed a similar analysis also for these phases, taking the dimer diagrams from [15]. We do not report here all the calculations, but make some useful comments. First of all we have checked that one can use the algorithm described in the previous subsection to determine the R-charges; this is an efficient and fast algorithm.

Model II, III and IV fit in the general analysis at the end of section 3. Model II and III of dP_3 have $F = 6$, $E = 14$, $V = 8$. They both have the same distribution of fields: there are all the fields that appeared in Model I with the same R-charges (5.15) plus two other fields: one has R-charge $a_3 + a_4 + a_5$ and the other $a_1 + a_2 + a_6$. Their contribution to the trial central charge a cancels:

$$(a_3 + a_4 + a_5 - 1)^3 + (a_1 + a_2 + a_6 - 1)^3 = 0 \tag{5.16}$$

because of (3.5). So the trial a charge to maximize is the same as in Model I.[13] Model IV of dP_3 has $F = 6$, $E = 18$, $V = 12$. There are all the fields appearing in Model I plus the six fields with R-charge: $a_3 + a_4 + a_5$, $a_1 + a_2 + a_6$, $a_5 + a_6 + a_1$, $a_2 + a_3 + a_4$, $a_1 + a_2 + a_3$, $a_4 + a_5 + a_6$. Again their contribution to the trial central charge cancels.

5.6 Orbifolds and singular horizons

In this subsection we deal with the problem of toric cones over non smooth five dimensional horizons; their toric diagram is characterized by the fact that some of its sides pass through integer points: let's call p the total number of such points on the sides. The global symmetries are now $d + p - 1$. So we have to add new variables to the a_i, $i = 1, \dots d$ if we want to find all the global charges. Let's call the new variables b_i, $i = 1, \dots p$.

For simplicity we shall work on a specific example: a particular realization of $L^{2,6;2,6}$ whose toric diagram and (p, q) web are drawn in figures 11 and 12. This example has $d = 4$ and $p = 4$. The double area of the toric diagram is $F = 8$

We have considered two toric phases of $L^{2,6;2,6}$. Their dimers are represented in figures 13 and 14 respectively. In fact it is not difficult to get the gauge theory by partial resolution of $C^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, by resolving the point of coordinates $(3, 0)$. The orbifold has 9 gauge groups and its gauge theory is described in [15]. The only way to get a theory with 8 gauge groups is by eliminating (any) one of the links in the dimer of $C^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Integrating out the massive fields one gets Model II, figure 14, which has $E = 22$, $V = 14$. Performing a Seiberg duality with respect to the gauge group corresponding to face E in

[13] Note that there may exist other parametrizations of trial R-charges. For example in Model III, one can find an equivalent distribution interchanging a_1 and a_4: this still satisfies the linear constraints from the vanishing of beta functions and conservation of superpotential. The expression of the trial central charge to maximize is different, but the results at the maximum, where $a_1 = a_4$, are the same. This is due to the high degree of symmetry of the toric diagram of dP_3.

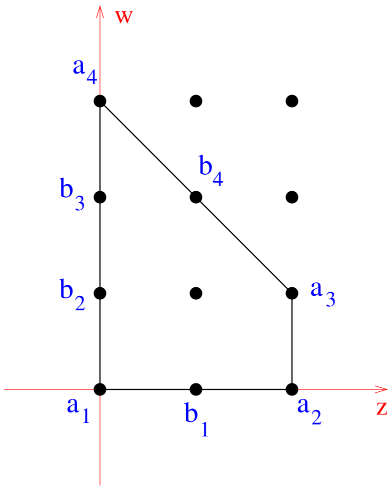

Figure 11: Toric diagram for $L^{2,6;2,6}$.

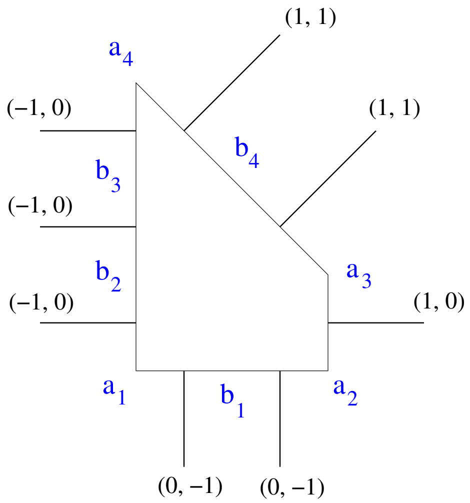

Figure 12: The (p, q) web for $L^{2,6;2,6}$.

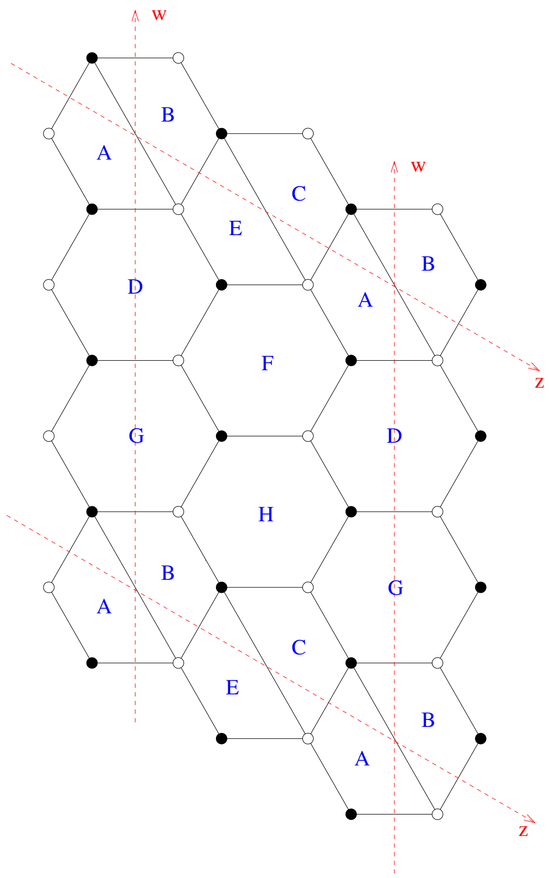

Figure 13: Model I for $L^{2,6;2,6}$.

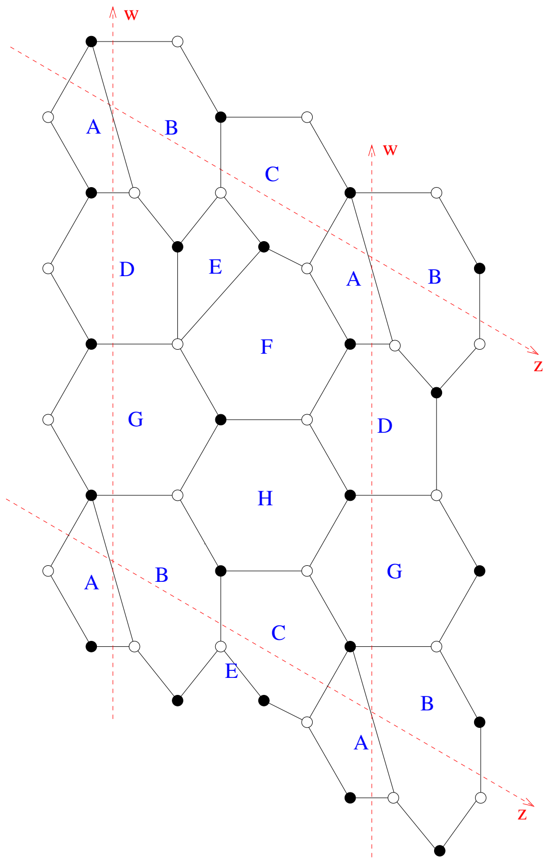

Figure 14: Model II for $L^{2,6;2,6}$.

the dimer of figure 14, one gets Model I for this theory, which has fewer fields: $E = 20$, $V = 12$.

We identify Model I with the toric phase with a minimal number of fields for which our formulae should work. In fact it is possible to extend the algorithm described in section 3 to extract multiplicities from the toric diagram. Now one should assign charge a_i to the d vertices V_i and b_j to the p integer points along the edges of P. Then the multiplicities are extracted using all the vectors of the (p,q) web as in figure 12. In our particular example one gets the fields:

$$\begin{array}{l} a_1\,,\quad a_1+b_1\,,\quad b_1+a_2\,,\quad b_1+a_2+a_3\,,\quad b_1+a_2+a_3+b_4\,,\quad a_2\,,\quad a_2+a_3\,,\\ a_2+a_3+b_4\,,\quad a_3\,,\quad a_3+b_4\,,\quad b_4+a_4\,,\quad b_4+a_4+b_3\,,\quad b_4+a_4+b_3+b_2\,,\quad a_4\,,\\ a_4+b_3\,,\quad a_4+b_3+b_2\,,\quad b_3+b_2+a_1\,,\quad b_3+b_2+a_1+b_1\,,\quad b_2+a_1\,,\quad b_2+a_1+b_1 \end{array} \tag{5.17}$$

all with multiplicity equal to one (the total number of fields is thus 20, as in Model I). Note that, differently from the case of a_i, there is no chiral field with charge, say, b_1, since the b_i are always included between parallel vectors (forming a parallelogram with area zero). Indeed it is not difficult to find a distribution of R-charges in the dimer configuration of Model I with these kinds of fields. Remember that the constraints are:

$$\sum_{i=1}^{d} a_i + \sum_{j=1}^{p} b_j = 2 \tag{5.18}$$

if we are dealing with R-charges, and

$$\sum_{i=1}^{d} a_i + \sum_{j=1}^{p} b_j = 0 \tag{5.19}$$

if we are dealing with global charges. The trial R-charge depends both on a_i and b_j, however we have verified in this case that the point that maximizes the central charge has all b_i equal to zero. We conjecture that this may be true in general. In practice one could have started with the (p,q) web drawn in figure 15 for $L^{2,6;2,6}$; this is simply built ignoring the fact that there are points on the sides of P: the vectors are not the primitive ones, but they have the same length as the vectors of P.

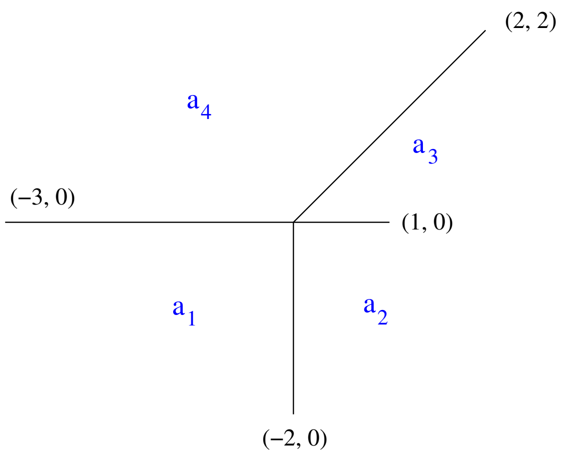

Figure 15: A rough version of the (p,q) web for $L^{2,6;2,6}$.

Using the usual method for multiplicities as in section 3 with the (p,q) web in figure 15, we get this table of multiplicities for the 20 fields:

$$\begin{array}{llllll} \text{R-charge:} & a_1 & a_2 & a_3 & a_4 & a_2+a_3 \\ \text{multiplicity:} & 6 & 2 & 2 & 6 & 4 \end{array} \tag{5.20}$$

to which (5.17) obviously reduces after setting $b_i = 0$. Then the a-maximization can be performed also keeping into account only the charges a_i and it is easy to check in this example that it reproduces the volumes of Z-minimization.

Let make also some comments about the generalization of the method described in subsection 5.4 for assigning (R-)charges. The multiplicities of perfect matchings associated with vertices are again equal to one. Then we assign to the corresponding perfect matching (R-)charge a_i. But in general there is more than one perfect matching corresponding to a certain point along a side of P. In Model I of the example at hand the multiplicities of perfect matchings corresponding to points b_1, b_2, b_3, b_4 are respectively 2, 3, 3, 2. Therefore for every point along the sides we can choose a particular perfect matching and give it (R-)charge b_i (and zero charge to all other perfect matchings). Then we can compute the charge of chiral fields as sums of charges of the perfect matchings to which they belong, as in subsection 5.4. In this way one always find R-charges (or global charges). However not all charges built in this way are linearly independent: this depends on the choice of perfect matchings. We verified in the case at hand that there are choices of perfect matchings for the b_i that allow to find all the 7 independent (R-)charges, some of them also reproducing the fields content given in (5.17).

The same conclusions hold for Model II of $L^{2,6;2,6}$. The number of fields now is 22: again a-maximization can be performed by setting to zero the b_i. We have all the fields appearing in table 5.20 plus one field with charge $a_1 + a_2$ and one with charge $a_3 + a_4$, so that the trial R-charge is the same as in Model I for the mechanism described at the end of section 3.

In conclusion in this subsection we have generalized our results to the case of non smooth horizon, checking in detail the algorithms on a particular example. This analysis deserves further study in order to verify whether it is true in the general case. In particular we guess that charges associated with points along the sides of the toric diagram are never relevant for a-maximization.

6. Conclusions

In this paper we computed the central charge and the R-charges of chiral fields for all the superconformal gauge theories living on branes at toric conical singularities. We also showed that the a-maximization technique [13] is completely equivalent to the volume minimization technique proposed in [14]. This, by itself, is an absolutely general check of the AdS/CFT correspondence, valid for all toric singularities.

In this general construction, something is obviously missing. We have now, using the tiling construction [15], a direct determination of the singularity associated with a given gauge theory. The inverse process is still incomplete: we can determine R-charges and multiplicities of fields but not the specific distribution of bi-fundamentals in the quiver theory. We are quite confident that, in the long period, the dimers technology will allow to define a one-to-one correspondence between CFTs and toric singularities.

It would be also interesting to derive the assignment of charges and multiplicities we propose here. A possible way of deriving it goes though mirror symmetry. It would be interesting to perform the analysis done in [19] in the general case. This analysis would probably teach us also about the many toric phases that are associated with the same superconformal gauge theory.

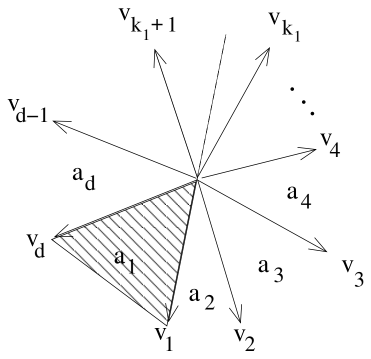

Figure 16: Summing multiplicities in C_1.

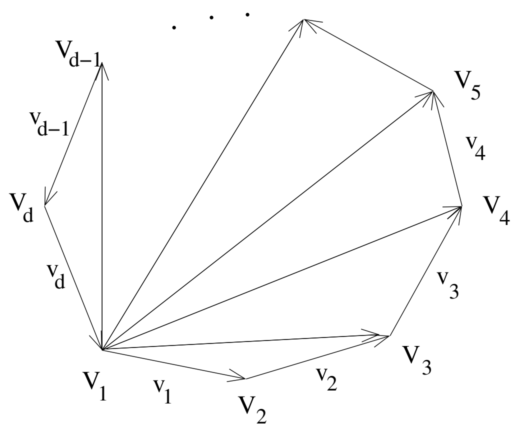

Figure 17: Computing the area of P.

Acknowledgments

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A. A useful formula

Let us define the sets C_h, $h = 1, 2 \dots d$ which are subsets of C: a couple (i, j) is in C_h iff the R-charge of the corresponding chiral field is a sum $a_{i+1} + a_{i+2} + \cdots + a_j$ containing a_h. In practice C_h is made up of all the couples (i, j) such that the region a_h in figure 2 is contained in the angle $\leq 180^o$ generated by v_i and v_j.

In this appendix we shall prove the useful formula

$$S_h \equiv \sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle| = \frac{V}{2} \tag{A.1}$$

where V is defined as:

$$V \equiv E - F \tag{A.2}$$

V is the number of vertices of the associated dimer configuration. Note that (when the convex polygon P has integer coordinates) equation (A.1) proves also that V is even. This agrees with the fact that there is an equal number of white and black vertices in the dimer configuration.

Given a vector v_j in the (p, q) web let us extend it (as in figure 16 for the case $j = 1$) and call v_{k_j} the vector in the (p, q) web just before this extension (moving in counter-clockwise direction). Note that:

$$\begin{aligned}
&|\langle v_j, v_{j+1}\rangle| + |\langle v_j, v_{j+2}\rangle| + \cdots + |\langle v_j, v_{k_j}\rangle| - \\
&\qquad - |\langle v_j, v_{j-1}\rangle| - |\langle v_j, v_{j-2}\rangle| - \cdots - |\langle v_j, v_{k_j+1}\rangle| = \\
&\qquad\qquad = \langle v_j, v_{j+1} + v_{j+2} + \cdots + v_{k_j} + v_{k_j+1} + \cdots + v_{j-1}\rangle \\
&\qquad\qquad = -\langle v_j, v_j\rangle \\
&\qquad\qquad = 0
\end{aligned} \tag{A.3}$$

where we have used that the sum of all v_i in the (p,q) web is zero. Remember that our indexes are always defined modulo d. Note that equation (A.3) is just the difference $S_{j+1} - S_j$, so we have proved that all S_j are equal.

To prove (A.1) we can choose $h = 1$ by a relabeling of vertices and sides (see figure 16). Let us consider the vector v_1 and write in the first line of a table all the multiplicities made up with v_1 (see below). We divide this line into two parts: on the left we write the pairs from $|\langle v_1, v_2\rangle|$ to $|\langle v_1, v_{k_1}\rangle|$ (those which do not contain a_1) and on the right the pairs from $|\langle v_1, v_d\rangle|$ to $|\langle v_1, v_{k_1+1}\rangle|$ (that contain a_1).[14] We repeat this procedure writing in the second line of the table all the pairs in C that contain v_2, again dividing the line into two parts: on the left the pairs from $|\langle v_2, v_3\rangle|$ to $|\langle v_2, v_{k_2}\rangle|$ and on the right the pairs from $|\langle v_2, v_1\rangle|$ to $|\langle v_2, v_{k_2+1}\rangle|$. We continue to fill in the lines with this ordering up to line k_1; in the remaining lines from $k_1 + 1$ to d we reverse the order in which we divide lines in a left and right part. For example line $k_1 + 1$ contains the multiplicities formed with v_{k_1+1} and we write on the left the pairs from $|\langle v_{k_1+1}, v_{k_1}\rangle|$ to $|\langle v_{k_1+1}, v_{k_{k_1+1}+1}\rangle|$ and on the right the pairs from $|\langle v_{k_1+1}, v_{k_1+2}\rangle|$ to $|\langle v_{k_1+1}, v_{k_{k_1+1}}\rangle|$: the idea is that all the pairs on the left do not contain a_1 whereas the pairs on the right may or may not contain a_1.

$$
\begin{array}{llllll}
|\langle v_1, v_2\rangle| & |\langle v_1, v_3\rangle| & \ldots\ |\langle v_1, v_{k_1}\rangle| & |\langle v_1, v_d\rangle| & |\langle v_1, v_{d-1}\rangle| & \ldots\ |\langle v_1, v_{k_1+1}\rangle| \\
|\langle v_2, v_3\rangle| & |\langle v_2, v_4\rangle| & \ldots\ |\langle v_2, v_{k_2}\rangle| & |\langle v_2, v_1\rangle| & |\langle v_2, v_d\rangle| & \ldots\ |\langle v_2, v_{k_2+1}\rangle| \\
\vdots & & & \vdots & & \\
|\langle v_{k_1}, v_{k_1+1}\rangle| & |\langle v_{k_1}, v_{k_1+2}\rangle| & \ldots\ |\langle v_{k_1}, v_{k_{k_1}}\rangle| & |\langle v_{k_1}, v_{k_1-1}\rangle| & |\langle v_{k_1}, v_{k_1-2}\rangle| & \ldots\ |\langle v_{k_1}, v_{k_{k_1}+1}\rangle| \\
|\langle v_{k_1+1}, v_{k_1}\rangle| & |\langle v_{k_1+1}, v_{k_1-1}\rangle| & \ldots\ |\langle v_{k_1+1}, v_{\ldots}\rangle| & |\langle v_{k_1+1}, v_{k_1+2}\rangle| & |\langle v_{k_1+1}, v_{k_1+3}\rangle| & \ldots\ |\langle v_{k_1+1}, v_{\ldots}\rangle| \\
\vdots & & & \vdots & & \\
|\langle v_d, v_{d-1}\rangle| & |\langle v_d, v_{d-2}\rangle| & \ldots\ |\langle v_d, v_{k_d+1}\rangle| & |\langle v_d, v_1\rangle| & |\langle v_d, v_2\rangle| & \ldots\ |\langle v_d, v_{k_d}\rangle|
\end{array}
$$

Note that the multiplicity associated with every pair of vectors v_i, v_j in C appears twice in the above table: once in line i and once in line j. Hence the total sum of multiplicities in the table is $2E$. But the sum of multiplicities in each line on the left equals the sum of multiplicities on the right in the same line because of (A.3). Hence the total sum of multiplicities on the right (or left) side of the table equals E. Moreover the pairs (i,j) of vectors with multiplicity $|\langle v_i, v_j\rangle|$ in the left side of this table do not belong to C_1. All the pairs of vectors in C_1 appear (twice) in the right side of the table; but on the right side there appear also pairs that do not belong to C_1: $|\langle v_2, v_1\rangle|$ in the second line, $|\langle v_3, v_2\rangle|$ and $|\langle v_3, v_1\rangle|$ in the second line, and so on. The total sum of such multiplicities is:

$$
\begin{aligned}
\langle v_1, v_2\rangle + \langle v_1 + v_2, v_3\rangle\ +\ & \cdots + \langle v_1 + v_2 + \cdots + v_{k_1-1}, v_{k_1}\rangle + \\
& + \langle v_{k_1+1}, v_{k_1+2} + v_{k_1+3} + \cdots + v_d\rangle + \cdots + \langle v_{d-1}, v_d\rangle = \\
& = \langle v_1, v_2\rangle + \langle v_1 + v_2, v_3\rangle + \cdots + \langle v_1 + v_2 + \cdots + v_{k_1-1}, v_{k_1}\rangle + \\
& + \langle v_1 + v_2 + \cdots + v_{k_1}, v_{k_1+1}\rangle + \langle v_1 + v_2 + \cdots + v_{d-2}, v_{d-1}\rangle \\
& = 2\mathrm{Area}(P) \\
& = F
\end{aligned}
\tag{A.4}
$$

where in the first equality we have used that the sum of all v_i is zero and the bilinearity and antisymmetry of the determinant. The sum of all multiplicities in the right side of the table above

[14]If there is a vector v_j lying just on the extension of v_1, the multiplicity $|\langle v_1, v_j\rangle|$ is zero and so it can be ignored.

that do not belong to C_1 is thus equal to the double area of P, see figure 17. The sum we had to compute is therefore:

$$S_1 \equiv \sum_{(i,j)\in C_1} |\langle v_i, v_j\rangle| = \frac{E-F}{2} = \frac{V}{2} \tag{A.5}$$

which is relation (A.1).

B. Charges

We now show that our proposed formula for extracting multiplicities of chiral fields from the toric diagram correctly gives U(1) baryon, flavor and R-charges with trace equal to zero. Let's start with a charge commuting with supersymmetry; as explained in section 3 it can be built by assigning charges a_i to chiral fields associated with vectors V_i of the fan with (3.8):

$$\sum_{i=1}^{d} a_i = 0\,. \tag{B.1}$$

Therefore we have $d-1$ global symmetries, 2 of which are flavor symmetries and the remaining $d-3$ are baryonic symmetries (remember that for non smooth horizons we have to consider also charges associated to integer points lying along the sides of the convex polygon P; the total sum of all charges associated to "fundamental" fields (B.1) must still be zero). The charge of a generic "composite" chiral field associated with the pair $(i,j) \in C$ is simply the sum $a_{i+1} + \cdots a_j$.

The trace of a generic U(1) global symmetry is thus:

$$\begin{aligned}
\operatorname{tr} \mathrm{U}(1) &= \sum_{(i,j)\in C} |\langle v_i, v_j\rangle| (a_{i+1} + a_{i+2} \cdots + a_j) \\
&= \sum_{h=1}^{d} a_h \sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle| = \frac{V}{2} \sum_{h=1}^{d} a_h = 0
\end{aligned} \tag{B.2}$$

where we have used that S_h in (A.1) does not depend on h.

Let us now turn to R-symmetry; to build the generic trial R-symmetry we have to associate a R-charge a_i to the chiral fields corresponding to divisors V_i (and also to fields corresponding to vertices along sides for non smooth horizons); the only difference with the global case is that now the sum must satisfy (3.5):

$$\sum_{i=1}^{d} a_i = 2\,. \tag{B.3}$$

The trace of a generic $\mathrm{U}(1)_R$ symmetry is now

$$\begin{aligned}
\operatorname{tr} \mathrm{U}(1)_R &= F + \sum_{(i,j)\in C} |\langle v_i, v_j\rangle|\ (a_{i+1} + a_{i+2} \cdots + a_j - 1) \\
&= F + \sum_{h=1}^{d} a_h \sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle| - \sum_{(i,j)\in C} |\langle v_i, v_j\rangle| \\
&= F + \frac{V}{2} \left(\sum_{h=1}^{d} a_h \right) - E = 0
\end{aligned} \tag{B.4}$$

where we have used equation (A.1). The term $F = 2\,\mathrm{Area}(P)$ comes from gauginos, since we know that the double area gives the number of gauge groups. This also shows that for gauge theories dual to toric geometries the trial R-charge always reduces to $a = 9/32\,\mathrm{tr}\,R^3$.

Let us now prove that the trace of cubic t'Hooft anomaly and mixed cubic anomaly for baryonic symmetries are always zero with the multiplicities and charges for chiral fields that we have conjectured in this paper. The vanishing of such anomalies is required by the AdS/CFT correspondence and is always true for the quiver gauge theories under consideration since the global baryonic symmetries are (the non anomalous) linear combinations of the U(1) part of the original gauge groups U(N) (after the AdS/CFT limit they generally become SU(N) gauge groups). But since we have only conjectured the multiplicities of chiral fields and a full algorithm for extracting the whole gauge theory from toric geometry is still lacking, the proof of zero cubic anomaly for baryonic symmetries is a non trivial check of our conjecture.

First of all recall that, as discovered in [9], the $d-1$ baryonic symmetries are simply the linear relations between the d generators of the toric fan V_i:[15] if $(a_1, a_2, \dots a_d)$ are the charges of a baryonic symmetry associated with chiral fields corresponding to the vectors V_i we have equation (3.4)

$$\sum_{i=1}^{d} a_i V_i = 0\,. \tag{B.5}$$

Knowing that V_i have first coordinate equal to 1, and that the other two components are the coordinates (x_i, y_i) of the vertices of P in the plane, the previous equation can also be restated by saying that $(a_1, a_2, \dots a_d)$ must satisfy (B.1), as all global symmetries, and moreover the constraint:

$$a_2 v_1 + a_3(v_1 + v_2) \cdots + a_d(v_1 + v_2 + \cdots v_{d-1}) = 0 \tag{B.6}$$

where we have started to compute the coordinates of the vertices of P from the first vertex (see figure 17), but one could have started from any other point in the plane of P because of (B.1). Note also that a basis for the two flavor symmetries orthogonal to the baryonic ones is given by the x and y coordinates of the vertices of P in the plane containing P referred to the barycenter of P, so that (B.1) holds.

So take now three different (or equal) baryonic symmetries: $(a_1, a_2, \dots a_d)$, $(a'_1, a'_2, \dots a'_d)$ and $(b_1, b_2, \dots b_d)$ all satisfying (B.1) and (B.6). To avoid writing too long formulae we will consider first the case when two symmetries are equal, say $a_i = a'_i$, and then we will extend our results to the general case. The mixed cubic t'Hooft anomaly with our formula for multiplicities becomes:

$$\begin{aligned}
\mathrm{tr}\,(\mathrm{U}(1)^a_B)^2\,\mathrm{U}(1)^b_B &= \sum_{(i,j)\in C} |\langle v_i, v_j\rangle|\,(a_{i+1} + a_{i+2} \cdots + a_j)^2 (b_{i+1} + b_{i+2} \dots b_j) \\
&= \sum_{h=1}^{d} b_h \left(\sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle|\,(a_{i+1} + a_{i+2} \cdots + a_j)^2 \right) \\
&\equiv \sum_{h=1}^{d} b_h c_h
\end{aligned} \tag{B.7}$$

where the coefficients c_h are defined by the last equality. We have to prove that the vector formed by c_h is orthogonal to a generic baryonic symmetry, that is that the vector of c_h is a linear combination of x and y coordinates of vertices of P up to some multiple of $(1, \dots, 1)$. So let's compute the differences:

$$\begin{aligned}
c_{j+1} - c_j &= |\langle v_j, v_{j+1}\rangle|\,(a_{j+1})^2 + |\langle v_j, v_{j+2}\rangle|\,(a_{j+1} + a_{j+2})^2 + \cdots + \\
&\quad + |\langle v_j, v_{k_j}\rangle|\,\left(a_{j+1} + a_{j+2} + \cdots + a_{k_j}\right)^2 -
\end{aligned}$$

[15] Again recall that for non smooth horizons one has to add to the set of V_i all the vectors in the fan arriving at the integer points along the sides of P.

$$
\begin{aligned}
&-\left|\langle v_j, v_{j-1}\rangle\right|(a_j)^2 - \left|\langle v_j, v_{j-2}\rangle\right|(a_j + a_{j-1})^2 - \cdots - \\
&-\left|\langle v_j, v_{k_j+1}\rangle\right|\left(a_j + a_{j-1} + \cdots + a_{k_j+2}\right)^2
\end{aligned}
$$

where there survive only the sum over pairs that contain a_{j+1} and do not contain a_j, minus the sum over pairs that contain a_j and do not contain a_{j+1}, since all other pairs cancel. The symbols k_j are defined as in appendix A. The previous equation can be rewritten as

$$
c_{j+1} - c_j = \langle v_j, T_j\rangle \tag{B.8}
$$

where T_j is the vector:

$$
\begin{aligned}
T_j &= v_{j+1}(a_{j+1})^2 + v_{j+2}(a_{j+1} + a_{j+2})^2 \cdots + v_{k_j}\left(a_{j+1} + a_{j+2} \cdots + a_{k_j}\right)^2 \\
&\quad + v_{j-1}(a_j)^2 + v_{j-2}(a_j + a_{j-1})^2 \cdots + v_{k_j+1}\left(a_j + a_{j-1} \cdots + a_{k_j+2}\right)^2 \\
&= v_{j+1}(a_{j+1})^2 + v_{j+2}(a_{j+1} + a_{j+2})^2 \cdots + v_{k_j}\left(a_{j+1} + a_{j+2} \cdots + a_{k_j}\right)^2 \\
&\quad + v_{k_j+1}\left(a_{j+1} + a_{j+2} \ldots a_{k_j+1}\right)^2 \cdots + v_{j-1}(a_{j+1} + a_{j+2} \cdots + a_{j-1})^2
\end{aligned} \tag{B.9}
$$

where in the last line we have reordered the sum and used that the sum of all a_i is zero (B.1).

Now we want to show that all vectors T_j are equal: $T_1 = T_2 \ldots = T_d \equiv T$; it is enough to prove that consecutive vectors T_j are equal and, by a relabeling of vectors and vertices, it is enough to prove this for, say T_1 and T_2. A straightforward computation then yields:

$$
\begin{aligned}
T_2 - T_1 &= v_3(a_3)^2 + v_4(a_3 + a_4)^2 + \cdots + v_d(a_3 + a_4 \cdots + a_d)^2 + \\
&\quad + v_1(a_3 + a_4 + \cdots + a_d + a_1)^2 - v_2(a_2)^2 - v_3(a_2 + a_3)^2 - \\
&\quad - v_4(a_2 + a_3 + a_4)^2 - \cdots - v_d(a_2 + a_3 \cdots + a_d)^2 \\
&= -a_2^2(v_2 + v_3 + \cdots + v_d) - 2a_2\left[v_3 a_3 + v_4(a_3 + a_4) \cdots + v_d(a_3 + a_4 \ldots a_d)\right] + \\
&\quad + v_1(a_2)^2 \\
&= 2a_2^2 v_1 - 2a_2\left[v_3 a_3 + v_4(a_3 + a_4) + \cdots + v_d(a_3 + a_4 \ldots a_d)\right] \\
&= -2a_2\left[v_3 a_3 + v_4(a_3 + a_4) + \cdots + v_d(a_3 + a_4 \ldots a_d) + v_1(a_3 + a_4 \cdots + a_d + a_1)\right] \\
&= -2a_2\big[a_1 v_1 + a_d(v_1 + v_d) + \\
&\qquad + a_{d-1}(v_1 + v_d + v_{d-1}) + \cdots + a_3(v_1 + v_d + v_{d-1} + \cdots + v_3)\big] \\
&= 0
\end{aligned} \tag{B.10}
$$

where we have used (B.1) and that the sum of v_i is zero. In the last step we have used that $(a_1, \ldots a_d)$ is a baryonic symmetry, since the last sum is one of the kind of (B.6), centered in the second vertex of the polygon P.

Now we get for the differences:

$$
\begin{aligned}
c_2 - c_1 &= \langle v_1, T\rangle \\
c_3 - c_1 &= (c_3 - c_2) + (c_2 - c_1) = \langle v_1 + v_2, T\rangle \\
&\ \vdots \\
c_d - c_1 &= \langle v_1 + v_2 \cdots + v_{d-1}, T\rangle
\end{aligned} \tag{B.11}
$$

and for the cubic t'Hooft anomaly of baryonic symmetries:

$$
\operatorname{tr}\left(\mathrm{U}(1)_B^a\right)^2 \mathrm{U}(1)_B^b = \sum_{h=1}^{d} b_h c_h
$$

$$
\begin{aligned}
&= c_1\left(b_1+b_2+b_3+\cdots+b_d\right)+\\
&\quad +b_2\left(c_2-c_1\right)+b_3\left(c_3-c_1\right)+\cdots+b_d\left(c_d-c_1\right)\\
&= \langle b_2 v_1+b_3\left(v_1+v_2\right)+\cdots+b_d\left(v_1+v_2\cdots+v_{d-1}\right), T\rangle\\
&= 0
\end{aligned}
\tag{B.12}
$$

where we have used that $(b_1, \ldots b_d)$ is a baryonic symmetry thus satisfying (B.1) and (B.6). It is easy to generalize to the case $a_i \neq a_i'$: the coefficients c_h are given now by:

$$
c_h = \sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle| \left(a_{i+1}+a_{i+2}\cdots+a_j\right)\left(a_{i+1}'+a_{i+2}'\cdots+a_j'\right) \tag{B.13}
$$

and one has to repeat all the steps leading to (B.10) keeping products of sums of a_i and a_i' instead of squares. It is to see that now (B.10) reads:

$$
\begin{aligned}
T_2-T_1 &= -a_2\big[a_1' v_1+a_d'\left(v_1+v_d\right)+a_{d-1}'\left(v_1+v_d+v_{d-1}\right)+\cdots+\\
&\qquad +a_3'\left(v_1+v_d+v_{d-1}+\cdots+v_3\right)\big]-\\
&\quad -a_2'\big[a_1 v_1+a_d\left(v_1+v_d\right)+\\
&\qquad +a_{d-1}\left(v_1+v_d+v_{d-1}\right)+\cdots+a_3\left(v_1+v_d+v_{d-1}\cdots+v_3\right)\big]\\
&= 0
\end{aligned}
\tag{B.14}
$$

so that one has to use that both a_i and a_i' are baryonic. The proof then proceeds as before (B.12). This concludes our proof for the cubic anomaly of baryonic symmetries:

$$
\operatorname{tr} \mathrm{U}(1)_B^a\ \mathrm{U}(1)_B^{a'}\ \mathrm{U}(1)_B^b = 0\,. \tag{B.15}
$$

C. Decoupling baryon charges in a-maximization

In this appendix we shall prove equation (4.15):

$$
\sum_{h=1}^{d} b_h \frac{\partial a}{\partial a_h}_{|a_i=f_i(x,y)} = 0 \tag{C.1}
$$

for every baryonic symmetry with charges b_i for the chiral fields associated to V_i. The functions $f_i(x,y)$ and $l_i(x,y)$ are defined as in (4.5) and (4.2):

$$
f_i = \frac{2\, l_i}{S}\,, \qquad l_i = \frac{\langle v_{i-1}, v_i\rangle}{A_{i-1}\, A_i} \tag{C.2}
$$

where we have defined the sum

$$
S \equiv \sum_{i=1}^{d} l_i \tag{C.3}
$$

and the double area of triangles in figure 3:

$$
A_i \equiv \langle r_i, v_i\rangle = \langle r_{i+1}, v_i\rangle\,. \tag{C.4}
$$

Remember that $r_{i+1} - r_i = v_i$. Note that l_i is positive inside the interior of P and diverges on the edges v_i and v_{i-1}.

We will need some useful relations among these quantities. In particular we can prove the vectorial identity:

$$
l_i\, r_i = \frac{v_{i-1}}{A_{i-1}} - \frac{v_i}{A_i}\,. \tag{C.5}
$$

In fact a straightforward computation gives

$$l_i\, r_i - \left(\frac{v_{i-1}}{A_{i-1}} - \frac{v_i}{A_i}\right) = \frac{\langle v_{i-1}, v_i\rangle r_i - v_{i-1}\langle r_i, v_i\rangle + v_i\langle r_i, v_{i-1}\rangle}{A_{i-1}\, A_i} \equiv \frac{N}{A_{i-1}\, A_i} \tag{C.6}$$

Then we get for the numerator $\langle N, v_i\rangle = \langle N, v_{i-1}\rangle = 0$. So N has to be parallel both to v_i and v_{i-1} which are two linearly independent vectors. Therefore $N = 0$ and we have proved (C.5).

By summing up (C.5) we get another important property:

$$\sum_{i=1}^{d} l_i r_i = 0 \tag{C.7}$$

which says that the l_i are (proportional to) the weights that should be put on the vertices V_i of P to keep it in equilibrium if we want to suspend it by the internal point B.

Equation (C.1) then reads

$$\sum_{h=1}^{d} b_h \frac{\partial a}{\partial a_h}_{|a_i=f_i(x,y)} = \frac{27}{32}\sum_{h=1}^{d} b_h d_h \tag{C.8}$$

where we have defined

$$d_h = \left(\sum_{(i,j)\in C_h} |\langle v_i, v_j\rangle| (a_{i+1} + a_{i+2} \cdots + a_j - 1)^2\right)_{|a_i=f_i(x,y)} \tag{C.9}$$

that comes from deriving (4.6) with respect to a_h. Note that (C.8) is, up to a constant factor, equal to tr $R^2 b$, with R the trial R symmetry and b the baryon charge. Note in fact the similarities with equation (B.7): the main difference here being that we are dealing with R-symmetry, so the constraint on a_i is (3.5), automatically implemented by the substitution (C.2).

Again the idea is to compute the differences $d_{j+1} - d_j$ and, similarly to (B.8), to rewrite them as

$$d_{j+1} - d_j = \langle v_j, \tilde{W}_j\rangle \tag{C.10}$$

where now the vector $\tilde{W}_j$ reads:

$$\begin{aligned}\tilde{W}_j \;=\; & \Big[v_{j+1}\,(a_{j+1} - 1)^2 + v_{j+2}\,(a_{j+1} + a_{j+2} - 1)^2 + \cdots + \\ & +v_{k_j}\,\big(a_{j+1} + a_{j+2} + \cdots + a_{k_j} - 1\big)^2 + v_{j-1}\,(a_j - 1)^2 + v_{j-2}\,(a_j + a_{j-1} - 1)^2 + \cdots + \\ & +v_{k_j+1}\,\big(a_j + a_{j-1} + \cdots + a_{k_j+2} - 1\big)^2\Big]_{|a_i=f_i}\end{aligned} \tag{C.11}$$

where the symbols k_j are defined as in appendix A. Performing the substitution (C.2) $a_i = f_i$ and taking the common denominator we get

$$\begin{aligned}S^2\tilde{W}_j \;=\; & v_{j+1}\,(l_{j+1} - l_{j+2} - l_{j+3} - \cdots - l_j)^2 + v_{j+2}\,(l_{j+1} + l_{j+2} - l_{j+3} - \cdots - l_j)^2 + \cdots + \\ & +v_{k_j}\,\big(l_{j+1} + l_{j+2} + \cdots + l_{k_j} - l_{k_j+1} \cdots - l_j\big)^2 + \\ & +v_{j-1}\,(l_j - l_{j-1} - l_{j-2} - \cdots - l_{j+1})^2 + v_{j-2}\,(l_j + l_{j-1} - l_{j-2} - \cdots - l_{j+1})^2 + \cdots + \\ & +v_{k_j+1}\,\big(l_j + l_{j-1} + \cdots + l_{k_j+2} - l_{k_j+1} - \cdots - l_{j+1}\big)^2 \\ =\; & v_{j+1}\,(l_{j+1} - l_{j+2} - l_{j+3} - \cdots - l_j)^2 + v_{j+2}\,(l_{j+1} + l_{j+2} - l_{j+3} - \cdots - l_j)^2 + \cdots + \\ & +v_{k_j}\,\big(l_{j+1} + l_{j+2} + \cdots + l_{k_j} - l_{k_j+1} - \cdots - l_j\big)^2 + \\ & +v_{k_j+1}\,\big(l_{j+1} + l_{j+2} + \cdots + l_{k_j+1} - l_{k_j+2} - \cdots - l_j\big)^2 + \cdots + \\ & +v_{j-1}\,(l_{j+1} + l_{j+2} + \cdots + l_{j-1} - l_j)^2\end{aligned} \tag{C.12}$$

where in the last step we have reordered the sum. For later convenience, let us add to $\tilde{W}_j$ two terms proportional to v_j defining the new vector W_j as:

$$\begin{aligned} S^2\, W_j \;=\; & v_{j+1}\,(l_{j+1}-l_{j+2}-l_{j+3}\cdots-l_j)^2+v_{j+2}\,(l_{j+1}+l_{j+2}-l_{j+3}-\cdots-l_j)^2+\cdots+ \\ & +v_{j-1}\,(l_{j+1}+l_{j+2}\cdots+l_{j-1}-l_j)^2+ \qquad \text{(C.13)} \\ & +v_j\,(l_{j+1}+l_{j+2}\cdots+l_{j-1}+l_j)^2-8S\frac{v_j}{A_j} \qquad \text{(C.14)} \end{aligned}$$

and because of antisymmetry of the determinant we still have:

$$d_{j+1}-d_j=\langle v_j, W_j\rangle\,. \tag{C.15}$$

We want to prove that all W_j are equal: $W_1=W_2\ldots=W_d\equiv W$. As in the previous appendix, it is enough to show the equality of consecutive W_j, W_{j+1}, and, up to a relabeling of indexes, it is enough to show that $W_2=W_1$. So let's compute the difference:

$$\begin{aligned} S^2(W_2-W_1) \;=\; & v_3\,(l_3-l_4-l_5\cdots-l_1-l_2)^2+v_4\,(l_3+l_4-l_5\cdots-l_1-l_2)^2+\cdots+ \\ & +v_1\,(l_3+l_4+l_5\cdots+l_1-l_2)^2+v_2\,(l_3+l_4+l_5\cdots+l_1+l_2)^2- \\ & -v_2\,(l_2-l_3-l_4-l_5\cdots-l_1)^2-v_3\,(l_2+l_3-l_4-l_5\cdots-l_1)^2- \\ & -v_4\,(l_2+l_3+l_4-l_5\cdots-l_1)^2\cdots-v_1\,(l_2+l_3+\cdots+l_d+l_1)^2- \\ & -8S\left(\frac{v_2}{A_2}-\frac{v_1}{A_1}\right) \\ =\; & 4l_2\,[v_2(l_3+l_4+l_5\cdots+l_1)+v_3(-l_3+l_4+l_5\cdots+l_1)+ \\ & +v_4(-l_3-l_4+l_5\cdots+l_1)\cdots+v_1(-l_3-l_4-l_5\cdots-l_1)]+ \\ & +8S\left(\frac{v_1}{A_1}-\frac{v_2}{A_2}\right) \end{aligned} \tag{C.16}$$

where in the last step we have computed the differences between factors with the same v_i keeping in consideration that each time only the term l_2 changes relative sign. Now we reorder the first term in the square bracket and we use equation (C.5) (with $i=2$) for the last term:

$$\begin{aligned} S^2(W_2-W_1) \;=\; & 4l_2\,[l_3\,(v_2-v_3-v_4-\cdots-v_1)+l_4\,(v_2+v_3-v_4-\cdots-v_1)+\cdots+ \\ & +l_1\,(v_2+v_3+v_4+\cdots+v_d-v_1)]+8Sl_2r_2 \\ =\; & 8l_2\,[l_3v_2+l_4\,(v_2+v_3)+\cdots+l_1\,(v_2+v_3+\cdots+v_d)]+8Sl_2r_2 \\ =\; & 8l_2\big[l_2r_2+l_3\,(r_2+v_2)+l_4\,(r_2+v_2+v_3)+\cdots+ \\ & +l_1\,(r_2+v_2+v_3+\cdots+v_d)\big]-8l_2r_2\left(\sum_{j=1}^{d} l_j\right)+8Sl_2r_2 \end{aligned} \tag{C.17}$$

where in the second equality we have used that $\sum_i v_i=0$, and in the third equality we have added and subtracted the same term. Now the last two terms cancel and, noting that $r_2+v_2+v_3\ldots v_{i-1}=r_i$ (look at figure 3) the sum in the square brackets becomes:

$$S^2(W_2-W_1)=8l_2\left(\sum_{j=1}^{d} l_jr_j\right)=0 \tag{C.18}$$

where we have used (C.7). Hence we conclude that $W_1=W_2\ldots=W_d\equiv W$. Now the conclusion of the proof of (C.1), that is $\sum_h b_hd_h=0$, proceeds as in (B.12) (with the appropriate substitutions $T\to W$, $c_h\to d_h$). In this step we use that b_i are baryonic. This concludes our proof.

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D. The equality of a and a^{MSY}

In this appendix we give a general proof of equation (4.16), that shows the agreement of the central charge a and the total volume even before maximization, once the substitution $a_i = f_i \equiv 2l_i/S$ has been performed.

Taking into consideration that $a = 9/32 \,\mathrm{tr}\, R^3$, the definition of a^{MSY} in (4.4) and equations (3.3), (4.2), what we have to prove is:

$$\mathrm{tr}\, R^3{}_{|a_i=f_i} = \frac{24}{S} \tag{D.1}$$

where S is the sum of l_i, as in the previous appendix (C.3).

In this appendix we will use the notation $b = (x, y)$ to indicate the point B in the plane of P (recall that the Reeb vector can be parametrized as $3(1, x, y)$). With a little abuse of notation, we will call V_i the coordinates (x_i, y_i) in the plane of P of the vertices V_i. Hence we have $v_i = V_{i+1} - V_i$ and $r_i = V_i - b$.

To simplify the calculation of $\mathrm{tr}\, R^3$, choose a point (x^0, y^0) in P, in general distinct from the "Reeb point" $b = (x, y)$. For every field in the quiver gauge theory (in the minimal toric phase described in section 3) associated with the pair $(i, j) \in C$ consider its R-charge:

$$a_{i,j} \equiv a_{i+1} + a_{i+2} \cdots + a_j \tag{D.2}$$

and perform the substitution $a_i = f_i(x, y)$; we get a rational function of (x, y). Perform the Taylor expansion of this function around the point (x^0, y^0) and denote with $\tilde{a}_{i,j}$ the truncation of this expansion up to linear terms in (x, y):

$$\begin{aligned} a_{i,j}(x,y) &= a_{i,j}(x_0, y_0) + \left(x_h - x_h^0\right) \frac{\partial}{\partial x_h} a_{i,j}(x_0, y_0) + O((x_h - x_h^0)^2) \\ &\equiv \tilde{a}_{i,j}(x, y) + O\left((x_h - x_h^0)^2\right) \end{aligned} \tag{D.3}$$

where x_h, $h = 1, 2$, is x or y.

We will use the fact that

$$\mathrm{tr}\, R^3 = \mathrm{tr}\, R^2 \tilde{R} \tag{D.4}$$

where $\tilde{R}$ stands for the vector of truncated R-charges $\tilde{a}_{i,j}$. In this formula and in the following we always understand the substitutions $a_i = f_i(x, y)$. To prove (D.4) note that, by multiplying by $2/S$ equation (C.7), we get:

$$\sum_{i=1}^{d} a_i r_i = 0\,, \qquad \Rightarrow \qquad \sum_{i=1}^{d} a_i V_i = 2b \tag{D.5}$$

since $r_i = V_i - b$. Note that this is just equation (2.86) in [14]. Deriving the last relation with respect to x and/or y we get:

$$\sum_{i=1}^{d} \left(\frac{\partial}{\partial x_h}\right)^k a_i V_i = 0\,, \qquad \text{if} \quad k \geq 2 \tag{D.6}$$

where the derivatives can be mixed in x, y and have total degree $k \geq 2$. In fact $b = (x, y)$ is linear in (x, y). Deriving instead the relation $\sum_i a_i = 2$ we get

$$\sum_{i=1}^{d} \left(\frac{\partial}{\partial x_h}\right)^k a_i = 0\,. \tag{D.7}$$

The two previous relations tell us that the derivatives of a_i with degree 2 or higher, calculated in any point (x^0, y^0), are baryonic symmetries: see equations (3.4) and (3.9). In the previous appendix we proved that for any baryonic symmetry $\mathrm{tr}\, R^2 B = 0$ for $a_i = f_i(x, y)$. Hence we have

$$\mathrm{tr}\, R^3 = \mathrm{tr}\, R^2 \left(\tilde{R} + \text{higher derivatives} \right) = \mathrm{tr}\, R^2 \tilde{R} \tag{D.8}$$

since the other terms in the Taylor expansion are derivatives with degree $k \geq 2$.

In the following we will choose (x_0, y_0) as the first vertex V_1 of P and we will calculate $\tilde{a}_{i,j}(x, y)$ in the point $b = (x, y)$. So we need to get the explicit expressions for the charges

$$\tilde{a}_i(x, y) = a_i(V_1) - r_1 \cdot \vec{\nabla} a_i(V_1)\,, \qquad \vec{\nabla} a_i = \left(\frac{\partial a_i}{\partial x}, \frac{\partial a_i}{\partial y} \right) \tag{D.9}$$

since $r_1 = V_1 - b = (x_0 - x, y_0 - y)$ and in the second term we have written the scalar product of this vector with the gradient of a_i. The charges of composite fields are obviously given by $\tilde{a}_{i,j} = \tilde{a}_{i+1} \cdots + \tilde{a}_j$.

Let us study first the behavior of $a_i(x, y) = 2l_i/S$ when $(x, y) = V_1 + tv_1$ approaches the point V_1 along the first side of P, $0 < t < 1$. Note that A_1 goes to zero, whereas the other areas A_i are strictly positive. Hence l_1 and l_2 goes to $+\infty$ and the other l_i remain finite. Hence all $a_i(x, y) = 2l_i/S$ different from a_1 and a_2 goes to zero, since they have a finite numerator and are divided by S which diverges. Performing the limit $(x, y) \to V_1 + tv_1$ for a_1 and a_2 we get:

$$\begin{aligned} a_1(V_1 + tv_1) &= \frac{2\langle v_d, v_1\rangle A_2}{A_2\langle v_d, v_1\rangle + A_d\langle v_1, v_2\rangle}_{|(x,y)=V_1+tv_1} = 2(1-t) \\ a_2(V_1 + tv_1) &= \frac{2\langle v_1, v_2\rangle A_d}{A_2\langle v_d, v_1\rangle + A_d\langle v_1, v_2\rangle}_{|(x,y)=V_1+tv_1} = 2t \end{aligned} \tag{D.10}$$

where we used $A_d = \langle v_d, tv_1\rangle$ and $A_2 = \langle (1-t)v_1, v_2\rangle$ when $(x, y) = V_1 + tv_1$. Note in particular that for $t = 0$, we obtain for the vertex V_1: $a_1(V_1) = 2$ and all other a_i equal to zero. Repeating this analysis on the last side v_d of P we obtain:

$$\begin{aligned} a_1(V_1 - tv_d) &= 2(1-t) \\ a_d(V_1 - tv_d) &= 2t \end{aligned} \tag{D.11}$$

and all other charges a_i equal to zero.

Deriving the previous relations with respect to t we obtain the gradient of the a_i along the sides v_1 and v_d of P:

$$\begin{array}{lll} v_1 \cdot \vec{\nabla} a_1(V_1) = -2 & v_1 \cdot \vec{\nabla} a_2(V_1) = 2 & v_1 \cdot \vec{\nabla} a_d(V_1) = 0 \\ v_d \cdot \vec{\nabla} a_1(V_1) = 2 & v_d \cdot \vec{\nabla} a_2(V_1) = 0 & v_d \cdot \vec{\nabla} a_d(V_1) = -2 \end{array} \tag{D.12}$$

and zero for all other charges different from a_d, a_1, a_2. Finally relation (C.5)

$$r_1 = \frac{v_d}{A_d\, l_1} - \frac{v_1}{A_1\, l_1} \tag{D.13}$$

allows to compute $\tilde{a}_i(x, y)$ from (D.9):

$$\left\{ \begin{array}{l} \tilde{a}_1 = 2 - 2\alpha - 2\beta \\ \tilde{a}_d = 2\alpha \\ \tilde{a}_2 = 2\beta \end{array} \right. \qquad \left\{ \begin{array}{l} \alpha \equiv \dfrac{A_1}{\langle v_d, v_1\rangle} \\ \beta \equiv \dfrac{A_d}{\langle v_d, v_1\rangle} \end{array} \right. \tag{D.14}$$

All other $\tilde{a}_i$ different from $\tilde{a}_d$, $\tilde{a}_1$, $\tilde{a}_2$ are zero. This fact, together with (D.4), allows to disentangle the complex combinatorics and to perform a straightforward, but quite long, computation of $\operatorname{tr} R^3$.

So we obtain:

$$\begin{aligned}
\operatorname{tr} R^3 &= \operatorname{tr} R^2 \tilde{R} = F + \sum_{(i,j)\in C} \langle v_i, v_j\rangle \left(a_{i,j}-1\right)^2 \left(\tilde{a}_{i,j}-1\right) \\
&= F - \sum_{(i,j)\in C} \langle v_i, v_j\rangle \left(a_{i,j}-1\right)\left(\tilde{a}_{i,j}-1\right) + \sum_{h=1}^{d} a_h \left(\sum_{(i,j)\in C_h} \langle v_i, v_j\rangle \left(a_{i,j}-1\right)\left(\tilde{a}_{i,j}-1\right)\right) \\
&\equiv F - \sum_{(i,j)\in C} \langle v_i, v_j\rangle \left(a_{i,j}-1\right)\left(\tilde{a}_{i,j}-1\right) + \sum_{h=1}^{d} a_h c_h
\end{aligned} \tag{D.15}$$

where c_h are defined by the last equality. With similar tricks as in previous appendices, we see that:

$$c_{j+1} - c_j = \langle v_j, T_j\rangle \tag{D.16}$$

where the vector T_j is:

$$\begin{aligned}
T_j &= \bar{T}_j + v_j - \frac{4}{S}\frac{v_j}{A_j} \\
\bar{T}_j &= v_{j+1}\left(a_{j+1}-1\right)\left(\tilde{a}_{j+1}-1\right)\cdots + v_{k_j}\left(a_{j+1}\cdots + a_{k_j}-1\right)\left(\tilde{a}_{j+1}\cdots+\tilde{a}_{k_j}-1\right) + \\
&\quad + v_{j-1}\left(a_j-1\right)\left(\tilde{a}_j-1\right)\cdots + v_{k_j+1}\left(a_j\cdots+a_{k_j+2}-1\right)\left(\tilde{a}_j\cdots+\tilde{a}_{k_j+2}-1\right) \\
&= v_{j+1}\left(a_{j+1}-1\right)\left(\tilde{a}_{j+1}-1\right) + v_{j+2}\left(a_{j+1}+a_{j+2}-1\right)\left(\tilde{a}_{j+1}+\tilde{a}_{j+2}-1\right) + \cdots + \\
&\quad + v_{j-1}\left(a_{j+1}+a_{j+2}\cdots+a_{j-1}-1\right)\left(\tilde{a}_{j+1}+\tilde{a}_{j+2}\cdots+\tilde{a}_{j-1}-1\right)
\end{aligned} \tag{D.17}$$

and the pieces proportional to v_j have been introduced for later convenience. For the difference of consecutive $\bar{T}_j$ we obtain:

$$\begin{aligned}
\bar{T}_2 - \bar{T}_1 &= v_3\left(a_3-1\right)\left(\tilde{a}_3-1\right) + v_4\left(a_3+a_4-1\right)\left(\tilde{a}_3+\tilde{a}_4-1\right) + \cdots + \\
&\quad + v_1\left(a_3+a_4\cdots+a_1-1\right)\left(\tilde{a}_3+\tilde{a}_4\cdots+\tilde{a}_1-1\right) - \\
&\quad - v_2\left(a_2-1\right)\left(\tilde{a}_2-1\right) - v_3\left(a_2+a_3-1\right)\left(\tilde{a}_2+\tilde{a}_3-1\right) - \cdots - \\
&\quad - v_d\left(a_2+a_3\cdots+a_d-1\right)\left(\tilde{a}_2+\tilde{a}_3\cdots+\tilde{a}_d-1\right) - \\
&= -a_2\tilde{a}_2\left[v_2+v_3\cdots+v_d\right] + v_2 a_2 + v_2\tilde{a}_2 - v_2 + v_1\left(a_2-1\right)\left(\tilde{a}_2-1\right) - \\
&\quad - a_2\left[v_3\left(\tilde{a}_3-1\right) + v_4\left(\tilde{a}_3+\tilde{a}_4-1\right)\cdots + v_d\left(\tilde{a}_3+\tilde{a}_4\cdots+\tilde{a}_d-1\right)\right] - \\
&\quad - \tilde{a}_2\left[v_3\left(a_3-1\right) + v_4\left(a_3+a_4-1\right)\cdots + v_d\left(a_3+a_4\cdots+a_d-1\right)\right] \\
&= -a_2\left[v_3\left(\tilde{a}_3-1\right) + v_4\left(\tilde{a}_3+\tilde{a}_4-1\right)\cdots + v_1\left(\tilde{a}_3+\tilde{a}_4\cdots+\tilde{a}_1-1\right)\right] - \\
&\quad - \tilde{a}_2\left[v_3\left(a_3-1\right) + v_4\left(a_3+a_4-1\right)\cdots + v_1\left(a_3+a_4\cdots+a_1-1\right)\right] + \\
&\quad + v_2 a_2 + v_2\tilde{a}_2 + v_1 - v_2 \\
&= -a_2\left[\tilde{a}_1 v_1 + \tilde{a}_d\left(v_1+v_d\right)\cdots + \tilde{a}_3\left(v_1+v_d\cdots+v_3\right)\right] - \\
&\quad - \tilde{a}_2\left[a_1 v_1 + a_d\left(v_1+v_d\right)\cdots + a_3\left(v_1+v_d\cdots+v_3\right)\right] + v_1 - v_2 \\
&= v_1 - v_2 + a_2\left(\sum_{i=1}^{d}\tilde{a}_i(V_i-V_2)\right) + \tilde{a}_2\left(\sum_{i=1}^{d} a_i(V_i-V_2)\right) \\
&= v_1 - v_2 - 2a_2 r_2 - 2\tilde{a}_2 r_2 = v_1 - v_2 - \frac{4}{S}\frac{v_1}{A_1} + \frac{4}{S}\frac{v_2}{A_2} - 2\tilde{a}_2 r_2
\end{aligned} \tag{D.18}$$

where in the last step we used (D.5) (which is also true for $\tilde{a}_i$, as one deduces from its Taylor expansion up to linear terms), $r_2 = V_2 - b$ and (C.5). By relabelling indices:

$$T_{j+1} - T_j = -2\tilde{a}_{j+1} r_{j+1}\,. \tag{D.19}$$

Note that

$$T_2 = T_3 \ldots = T_{d-1} \; = \; T_1 - 2\tilde{a}_2 r_2 \tag{D.20}$$

since $\tilde{a}_i$ are zero for $i = 3, 4, \ldots d-1$. We obtain then

$$\begin{aligned}
\sum_{h=1}^{d} a_h c_h \; &= \; c_1 \left(a_1 + a_2 \cdots + a_d\right) + a_2 \left(c_2 - c_1\right) \cdots + a_d \left(c_d - c_1\right) \\
&= \; 2c_1 + a_2 \langle v_1, T_1 \rangle + a_3 \left(\langle v_1, T_1 \rangle + \langle v_2, T_2 \rangle\right) \cdots + a_d \left(\langle v_1, T_1 \rangle \cdots + \langle v_{d-1}, T_{d-1} \rangle\right) \\
&= \; 2c_1 + \langle a_2 v_1 + a_3 \left(v_1 + v_2\right) \cdots + a_d \left(v_1 + v_2 \ldots v_d\right), T_1 \rangle - \\
&\quad -2\tilde{a}_2 \langle a_3 v_2 + a_4 \left(v_2 + v_3\right) \cdots + a_d \left(v_2 + v_3 \cdots + v_{d-1}\right), r_2 \rangle \\
&= \; 2c_1 + \langle \sum_{i=1}^{d} a_i \left(V_i - V_1\right), T_1 \rangle - 2\tilde{a}_2 \langle \sum_{i=1}^{d} a_i \left(V_i - V_2\right), r_2 \rangle + 2\tilde{a}_2 \langle \left(V_1 - V_2\right) a_1, r_2 \rangle \\
&= \; 2c_1 - 2\langle r_1, T_1 \rangle + 4\tilde{a}_2 \langle r_2, r_2 \rangle - 2\tilde{a}_2 a_1 \langle v_1, r_2 \rangle \\
&= \; 2c_1 - 2\langle r_1, T_1 \rangle + 4a_1 \frac{A_d A_1}{\langle v_d, v_1 \rangle} = 2c_1 - 2\langle r_1, T_1 \rangle + \frac{4a_1}{l_1} \\
&= \; 2c_1 - 2\langle r_1, T_1 \rangle + \frac{8}{S}
\end{aligned} \tag{D.21}$$

where we have used the explicit expression (D.14) for $\tilde{a}_2$, and performed the substitution $a_1 = 2l_1/S$.

From the definition we now compute:

$$\begin{aligned}
\bar{T}_1 \; &= \; v_2 \left(a_2 - 1\right)\left(\tilde{a}_2 - 1\right) + v_3 \left(a_2 + a_3 - 1\right)\left(\tilde{a}_2 + \tilde{a}_3 - 1\right) + \cdots + \\
&\quad + v_d \left(a_2 + a_3 + \cdots + a_d - 1\right)\left(\tilde{a}_2 + \tilde{a}_3 + \cdots + \tilde{a}_d - 1\right) \\
&= \left(\tilde{a}_2 - 1\right)\left[v_2 \left(a_2 - 1\right) + v_3 \left(a_2 + a_3 - 1\right) + \cdots + v_d \left(a_2 + \cdots + a_d - 1\right)\right] + \tilde{a}_d v_d (1 - a_1) \\
&= \left(\tilde{a}_2 - 1\right)\left[-\left(v_2 + v_3 \cdots + v_d\right) - \sum_{i=1}^{d} a_i \left(V_i - V_1\right)\right] + \tilde{a}_d v_d (1 - a_1) \\
&= \left(\tilde{a}_2 - 1\right)\left(v_1 + 2r_1\right) + \tilde{a}_d v_d (1 - a_1)
\end{aligned} \tag{D.22}$$

and hence

$$\begin{aligned}
\langle r_1, T_1 \rangle \; &= \; \left\langle r_1, \left(\tilde{a}_2 - 1\right)\left(v_1 + 2r_1\right) + \tilde{a}_d v_d \left(1 - a_1\right) + v_1 - \frac{4}{S}\frac{v_1}{A_1} \right\rangle \\
&= \; \tilde{a}_2 \langle r_1, v_1 \rangle + \tilde{a}_d \left(1 - a_1\right) \langle r_1, v_d \rangle - \frac{4}{S\, A_1} \langle r_1, v_1 \rangle \\
&= \; \frac{2A_d A_1}{\langle v_d, v_1 \rangle} + \frac{2A_d A_1}{\langle v_d, v_1 \rangle}(1 - a_1) - \frac{4}{S} = \frac{2}{l_1}\left(2 - \frac{2l_1}{S}\right) - \frac{4}{S} \\
&= \; \frac{4}{l_1} - \frac{8}{S}
\end{aligned} \tag{D.23}$$

where again we have used the explicit expressions for $\tilde{a}_i$ in (D.14) and the substitution $a_1 = 2l_1/S$. Collecting pieces together we obtain for (D.21):

$$\sum_{h=1}^{d} a_h c_h = 2c_1 + \frac{24}{S} - \frac{8}{l_1}\,. \tag{D.24}$$

Going back to (D.15) we obtain:

$$\operatorname{tr} R^3 \; = \; F - \sum_{(i,j) \in C} \langle v_i, v_j \rangle \left(a_{i,j} - 1\right)\left(\tilde{a}_{i,j} - 1\right) + \sum_{h=1}^{d} a_h c_h$$

$$= F + \sum_{(i,j)\in C} \langle v_i, v_j \rangle (a_{i,j} - 1) - \sum_{(i,j)\in C} \langle v_i, v_j \rangle (a_{i,j} - 1)\, \tilde{a}_{i,j} + \sum_{h=1}^{d} a_h c_h$$

$$= - \sum_{(i,j)\in C} \langle v_i, v_j \rangle (a_{i,j} - 1)\, \tilde{a}_{i,j} + \sum_{h=1}^{d} a_h c_h \tag{D.25}$$

where in the last step we used $\operatorname{tr} R = 0$ (B.4).

Let us expand the first term in the previous equality; we use the explicit form (D.14) of the $\tilde{a}_i$ putting in evidence the factors of 2, α and β:

$$\begin{aligned}\sum_{(i,j)\in C} \langle v_i, v_j \rangle (a_{i,j} - 1)\, \tilde{a}_{i,j} = \; & 2 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1) + \\ & +2\alpha \left(\sum_{(i,j)\in (C_d - C_1)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) - \\ & -2\alpha \left(\sum_{(i,j)\in (C_1 - C_d)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) + \\ & +2\beta \left(\sum_{(i,j)\in (C_2 - C_1)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) - \\ & -2\beta \left(\sum_{(i,j)\in (C_1 - C_2)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) .\end{aligned} \tag{D.26}$$

Expanding the factor $2c_1$ in (D.24) we obtain:

$$\begin{aligned} 2c_1 &= 2 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1)(\tilde{a}_{i,j} - 1) \\ &= -2 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1) + 2 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1)\, \tilde{a}_{i,j} \\ &= -2 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1) + 4 \sum_{(i,j)\in C_1} \langle v_i, v_j \rangle (a_{i,j} - 1) - \\ & -4\alpha \left(\sum_{(i,j)\in (C_1 - C_d)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) - 4\beta \left(\sum_{(i,j)\in (C_1 - C_2)} \langle v_i, v_j \rangle (a_{i,j} - 1) \right) \end{aligned} \tag{D.27}$$

Then equation (D.25), using (D.24), (D.26), (D.27), becomes:

$$\begin{aligned} \operatorname{tr} R^3 = & -2\alpha \left(\sum_{(i,j)\in (C_1 - C_d)} + \sum_{(i,j)\in (C_d - C_1)} \right) \langle v_i, v_j \rangle (a_{i,j} - 1) - \\ & -2\beta \left(\sum_{(i,j)\in (C_1 - C_2)} + \sum_{(i,j)\in (C_2 - C_1)} \right) \langle v_i, v_j \rangle (a_{i,j} - 1) + \frac{24}{S} - \frac{8}{l_1} \end{aligned} \tag{D.28}$$

and it is easy to compute:

$$\begin{aligned} & \left(\sum_{(i,j)\in (C_1 - C_d)} + \sum_{(i,j)\in (C_d - C_1)} \right) \langle v_i, v_j \rangle (a_{i,j} - 1) = \\ & \qquad = \langle v_d, v_1 \rangle (a_1 - 1) + \langle v_d, v_2 \rangle (a_1 + a_2 - 1) + \cdots + \end{aligned}$$

$$
\begin{aligned}
&+\langle v_d, v_{k_d}\rangle \left(a_1 + a_2 + \cdots + a_{k_d} - 1\right) - \\
&-\langle v_d, v_{d-1}\rangle \left(a_d - 1\right) - \cdots - \langle v_d, v_{k_d+1}\rangle \left(a_d + a_{d-1} + \cdots + a_{k_d+2} - 1\right) \\
=\;& \langle v_d, v_1 \left(a_1 - 1\right) + v_2 \left(a_1 + a_2 - 1\right) + \cdots + v_{d-1} \left(a_1 + a_2 + \cdots + a_{d-1} - 1\right)\rangle \\
=\;& \langle v_d, -\left(v_1 + v_2 + \cdots + v_{d-1}\right) + a_1 \left(v_1 + v_2 + \cdots + v_{d-1}\right) + \\
&+a_2 \left(v_2 + \cdots + v_{d-1}\right) + \cdots + a_{d-1} v_{d-1} \\
=\;& \left\langle v_d, -\sum_{i=1}^{d} a_i \left(V_i - V_d\right)\rangle = 2\langle v_d, r_d \right\rangle \\
=\;& -2A_d
\end{aligned} \tag{D.29}
$$

and similarly, with the opportune changes in the indexes:

$$
\left(\sum_{(i,j)\in(C_2-C_1)} + \sum_{(i,j)\in(C_1-C_2)} \right) \langle v_i, v_j\rangle \left(a_{i,j} - 1\right) = -2A_1 \tag{D.30}
$$

Finally equation (D.28) becomes:

$$
\begin{aligned}
\operatorname{tr} R^3 &= \frac{24}{S} - \frac{8}{l_1} - 2\frac{A_1}{\langle v_d, v_1\rangle} \left(-2A_d\right) - 2\frac{A_d}{\langle v_d, v_1\rangle} \left(-2A_1\right) \\
&= \frac{24}{S}
\end{aligned} \tag{D.31}
$$

and this concludes our proof.

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